Nonlinear Simulation of Drift Wave Turbulence

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The Australian National University

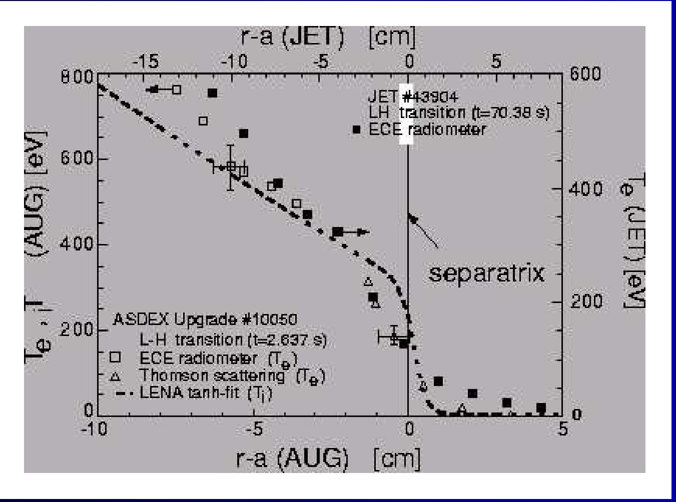
Ryusuke NUMATA, R. Ball, R.L. Dewar

ryusuke.numata@anu.edu.au

Low to High Confinement Transtion

In many magnetically confined fusion experiments like tokamaks and stellarators, plasma may undergo a spontaneous transition to a turbulence suppressed regime. Such transitions from a low-confinement (L) mode to a high-confinement (H) mode are known as L-H transitions.

The H-mode is characterized by steep gradients in density and temperature at the plasma edge.



[http://www.ipp.mpg.de/]

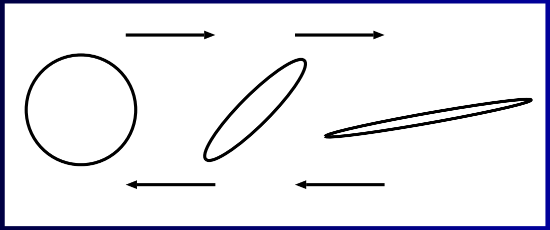
Transport Reduction by Zonal Flow

From theoretical and experimental works, it is now widely believed that stable coherent structures such as shear flows suppress cross-field turbulent transport and leads to the confinement improvement.
Random walk argument gives the diffusion coefficient

$$D \sim \frac{\Delta x^2}{\Delta t} \tag{1}$$

Collisional transport in a uniform magnetic field: $D \sim \nu \rho_{\mathrm{L}}^2$

Convective cell (a specific low-freq. eigenmode of the 2d fluid) $D \sim \frac{T}{eB} \frac{e\Delta\varphi}{T}$: (Bohm diffusion)

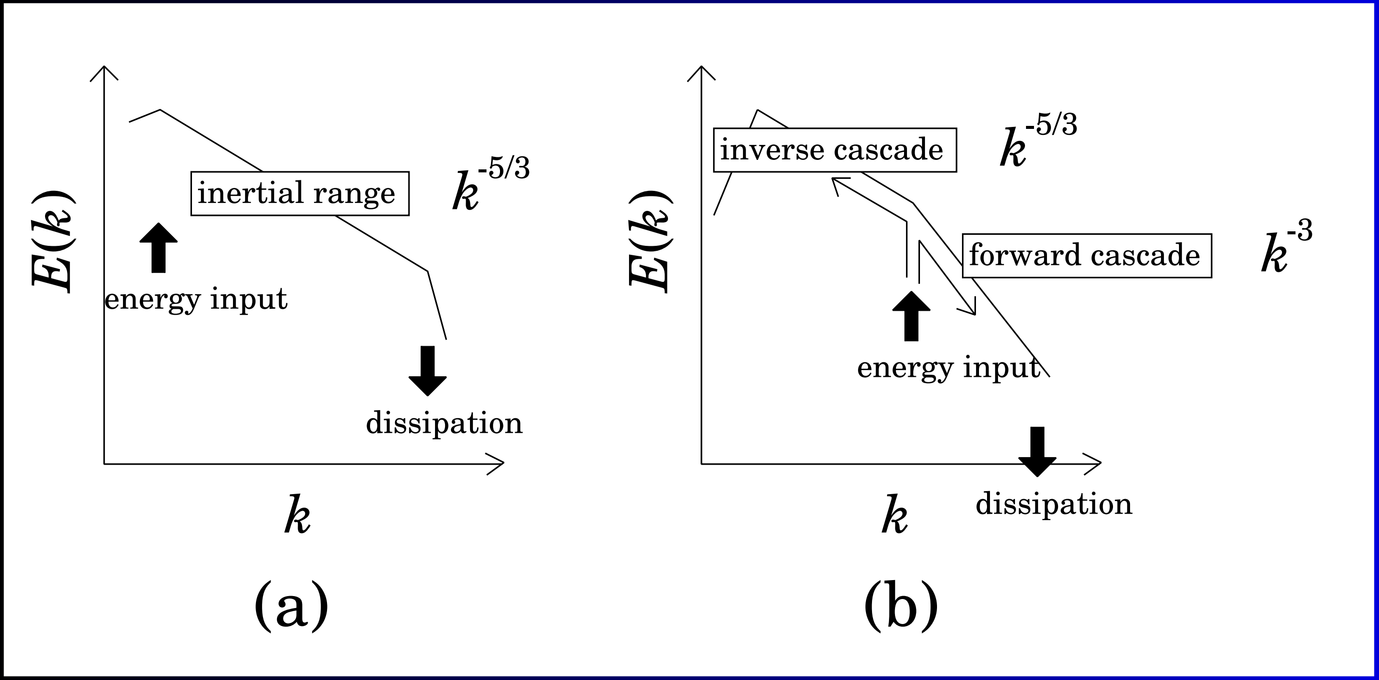


Interaction between a zonal flow and turbulent fluctuations is a key physics to understand the L-H transition.

Quasi Two Dimensional Flow

In two dimensional fluids the net upscale energy flux from small scale turbulent modes to create large scale coherent structures can dominate the classical Kolmogorov cascade to dissipative scales.

In the presence of the strong magnetic field, electrostatic fluctuations are confined almost in two-dimensionl plane perpendicular to the magnetic field.



Low Dimensional Dynamical Model

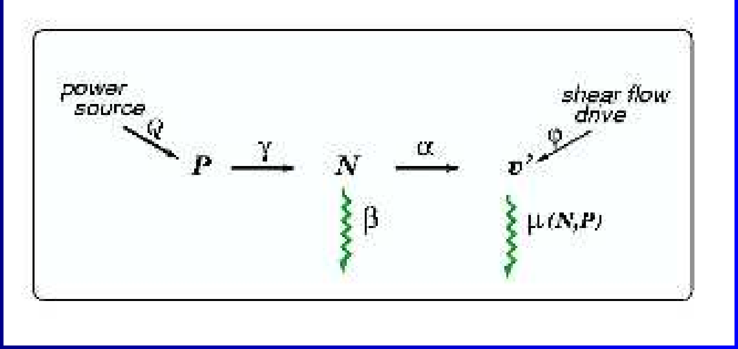
A low dimensional dynamical model is a reduction of a fluid system (infinite degree of freedom) to a system of rate equations of some macroscopic variables described by coupled ODEs (few degree of freedom).

Recently, Ball *et al*. (2002) derived a low dim. model for confinement transitions by integrating the reduced MHD equations. The model consists of three macroscopic state variables: P is the potential energy production, N is the turbulent kinetic energy, F is the shear flow kinetic energy,

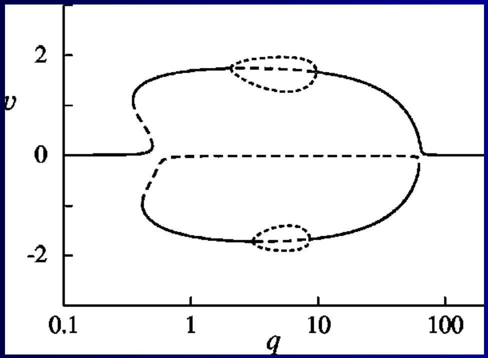
$$\varepsilon \frac{\mathrm{d}P}{\mathrm{d}t} = q - \gamma P N \quad (2)$$

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \gamma P N - \alpha F N - \beta N^2 \quad (3)$$

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \alpha F N - \mu(P,N)F + \varphi F^{1/2} \quad (4)$$



Bifurcation and Singularity Theory



Bifurcation and singularity theory of the derived dynamical model predicts ...

- Symmetry breaking term φ from drag (friction) force dissolves the pitchfork bifurcation.
- Hysteretic transition to a high confinement state.
- Spontaneous reversal of shear flow direction.
- Supersuppresion of turbulence at low power input.

The model will provide an economical tool to predict transitions over parameter space when validated against numerical simulation or real experimental data.

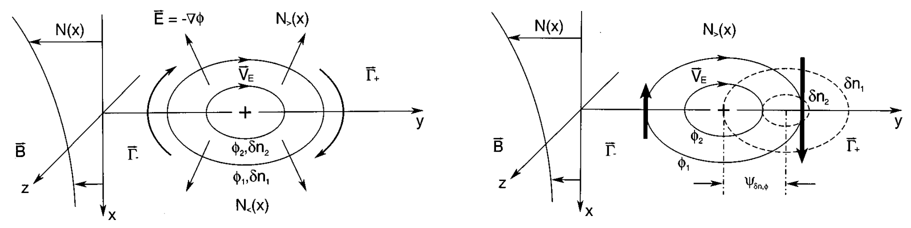
Drift Wave Mechanism

Tokamak edge fluctuations are often ascribed to drift wave turbulence.

Drift waves occur universally in magnetized plasmas producing the dominant mechanism for the transport of particles, energy, and momentum across magnetic field lines.

Density and temperature gradients in magnetized plasma give rise to electron and ion diamagnetic currents across the magnetic field. The drift velocity associated with these currents give rise to collective oscillations called drift wave.

In the presence of the resistivity, etc, the potential and the density is out of phase, this leads instability of damping $\longrightarrow$ turbulence



drift wave mechanism showing ExB convection
in a nonuniform magnetized plasma

Drift Wave Ordering and Assumptions

To derive the drift wave equations, let us consider an electrostatic fluctuations much slower than the ion cyclotron frequency ω_{ci} in a magnetized (with magnetic field $B_0 \nabla z$) and inhomogeneous plasma. If the parallel phase velocity ω / k_z lies between the electron and ion thermal speed, the resonant wave-particle interaction is small and a linear wave is known to exist in such a plasma. The fluctuations, or drift wave propagate with a speed approximately equal to the electron diamagnetic drift velocity.

$$\epsilon = \frac{1}{\omega_{\mathrm{ci}}} \frac{\partial}{\partial t} \sim \frac{1}{k_z v_{T_{\mathrm{e}}}} \frac{\partial}{\partial t} \sim \rho_{\mathrm{s}} \left| \nabla \left(\ln \frac{n_0}{B_0} \right) \right| \sim \frac{|\zeta|}{\omega_{\mathrm{ci}}} \tag{5}$$

where $\boldsymbol{\zeta} = \nabla \times \boldsymbol{V}_{\mathrm{i}}$ is the ion vorticity, $\rho_{\mathrm{s}} = \left(\frac{T_{\mathrm{e}}}{m_{\mathrm{i}}} \right)^{1/2} / \omega_{\mathrm{ci}}$ is the ion sound Larmor radius. We also assume that the electron temperature is much larger than the ion temperature, and use cold ion approximation.

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Resistive Drift Wave Model

In the drift wave ordering, ion and electron motion perpendicular to the magnetic field can be described by the $\boldsymbol{E} \times \boldsymbol{B}$ drift, the polarization drift, and the diamagnetic drift,

$$\boldsymbol{V}_{\perp,\mathrm{i}} = \epsilon \boldsymbol{V}_E + \epsilon^2 \boldsymbol{V}_\mathrm{p} = \frac{-\nabla\varphi \times \nabla z}{B_0} + \frac{\mathrm{d}}{\mathrm{d}t}(-\nabla\varphi), \tag{6}$$

$$\boldsymbol{V}_{\perp,\mathrm{e}} = \epsilon \boldsymbol{V}_E + \epsilon \boldsymbol{V}_\mathrm{d} = \frac{-\nabla\varphi \times \nabla z}{B_0} + \frac{\nabla p_\mathrm{e} \times \nabla z}{enB_0}, \tag{7}$$

where φ is the electrostatic potential, and $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \boldsymbol{V}_E \cdot \nabla$ is the $\boldsymbol{E} \times \boldsymbol{B}$ convective derivative. The electron continuity equation gives

$$\frac{\mathrm{d}}{\mathrm{d}t} n = \frac{\mathrm{d}}{\mathrm{d}t}(n_0 + n_1) = \frac{1}{e}\partial_z j_z, \tag{8}$$

and the quasi-neutrality demands $\nabla \cdot (n\boldsymbol{V}_\mathrm{i} - n\boldsymbol{V}_\mathrm{e}) = 0$ which leads

$$\frac{mn}{B_0}\frac{\mathrm{d}}{\mathrm{d}t}\nabla^2\varphi = B_0 \partial_z j_z. \tag{9}$$

This equation is equivalent to the ion vorticity equation.

Resistive Drift Wave Model

We assume that ion inertia along the magnetic field is negligible. Then the parallel current is determined by the Ohm's law (electron momentum equation),

$$\boldsymbol{E} + \frac{1}{en}\nabla p_{\mathrm{e}} = \eta \boldsymbol{j}. \tag{10}$$

If parallel heat conductivity is sufficiently large, electron may be treated as an isothermal fluid $p_{\mathrm{e}} = nT_{\mathrm{e}}$. The parallel current is obtained to be

$$j_z = -\frac{1}{\eta}\partial_z\left(\varphi - \frac{T_{\mathrm{e}}}{e}\ln n\right) \tag{11}$$

Substiting j_z into the continuity and vorticity equations, and normalizing variables as

$$x \to \rho_{\mathrm{s}} x, \;\; \omega_{\mathrm{ci}} t \to t, \;\; e\varphi/T_{\mathrm{e}} \to \varphi, \;\; n_1/n_0 \to n, \tag{12}$$

we finally obtain the so-called Hasegawa-Wakatani (HW) equations for the drift wave turbulence.

Hasegawa-Wakatani Equations

$$\frac{\partial}{\partial t}\zeta + \{\varphi, \zeta\} = \alpha(\varphi - n) \tag{13}$$

$$\frac{\partial}{\partial t}n + \{\varphi, n\} = \alpha(\varphi - n) - \kappa\frac{\partial \varphi}{\partial y} \tag{14}$$

where $\{a, b\} \equiv \partial_x a \partial_y b - \partial_y a \partial_x b$ is the Poisson bracket, $\zeta = \nabla^2 \varphi$ is the vorticity. The HW equations have two parameters;

- $\kappa \equiv -\partial_x \ln n_0$ is the scale of the background density profile. κ is assumed to be constant, which means exponential profile $n_0 \propto \exp(-\kappa x)$.
- $\alpha \equiv -\frac{T_{\mathrm{e}}}{\eta n_0 \omega_{\mathrm{ci}} e^2}\partial_z^2$ is the operator which describes parallel electron behavior. In the two-dimension model, α is reduced to be an parameter defined by $\alpha = \frac{T_{\mathrm{e}} k_z^2}{\eta n_0 \omega_{\mathrm{ci}} e^2}$ for a single parallel mode number k_z.

Hasegawa-Wakatani Equations

The HW model spans two limit according to the adiabaticity parameter.

Adiabatic limit $\alpha \to \infty$ The HW equations reduced to the Hasegawa-Mima (HM) or Charney-Obukhov equation,

$$\frac{\partial}{\partial t}(\varphi - \zeta) - \{\varphi, \zeta\} + \kappa \frac{\partial \varphi}{\partial y} = 0. \tag{15}$$

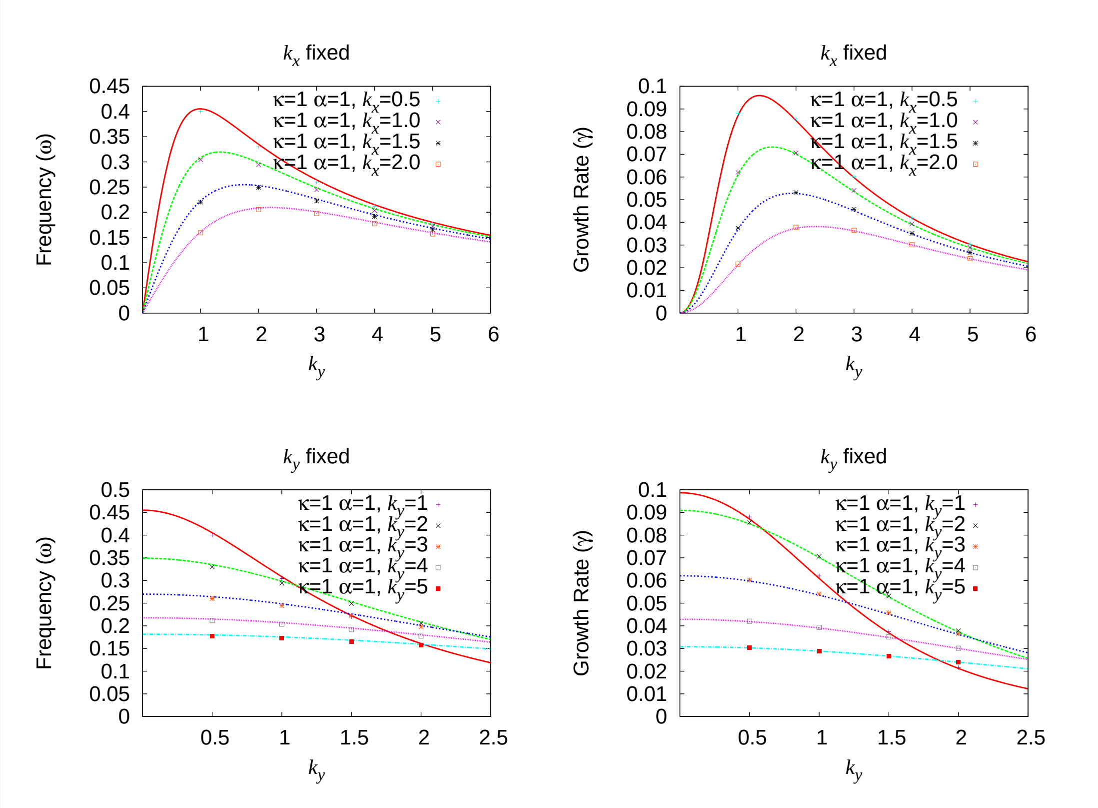
Electron moves fast enough along the field line to obey the Boltzmann relation,

$$n = n_0(x) \exp\left(\frac{e\varphi}{T_\mathrm{e}}\right). \tag{16}$$

Hydrodynamic limit $\alpha \to 0$ Vorticity evolution decouples with density. The equations become the Euler equation and a passive scalar equation for the density perturbation.

Linear Dispersion Relation

$$\omega + \mathrm{i}\gamma = \mathrm{i}\frac{b}{2}\left[(1-\frac{4\mathrm{i}\omega_*}{b})^{1/2}-1\right],\quad b=\alpha\frac{1+k^2}{k^2},\quad \omega_* = \kappa\frac{k_y}{1+k^2} \qquad (17)$$



Top: k_y dependence for fixed k_x

Bottom: k_x dependence for fixed k_y

In the adiabatic limit, the dispersion relation becomes $\omega = \omega_*$, while it has a finite γ (unstable) for non adiabatic case.

$k_y \sim 1$, $k_x \sim 0$ mode is most unstable.

Conservation Laws

The system has two dynamical invariants: the total energy E and the generalized enstrophy W, defined as follows,

$$E = \frac{1}{2}\int (n^2 + |\nabla\varphi|^2)\mathrm{d}\boldsymbol{x}, \quad W = \frac{1}{2}\int (n-\zeta)^2 \mathrm{d}\boldsymbol{x}. \tag{18}$$

where $\mathrm{d}\boldsymbol{x} = \mathrm{d}x\mathrm{d}y$. These quantities evolve with time as,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \Gamma_n - \Gamma_\alpha, \quad \frac{\mathrm{d}W}{\mathrm{d}t} = \Gamma_n \tag{19}$$

where

$$\Gamma_n = -\kappa \int n \frac{\partial \varphi}{\partial y}\mathrm{d}\boldsymbol{x}, \quad \Gamma_\alpha = -\alpha \int (n-\varphi)^2 \mathrm{d}\boldsymbol{x}. \tag{20}$$

In the inertial range, a dual cascade of energy and enstrophy will be found as in the usual two-dimensional fluid turbulence. Energy may condensate in a large scale structure. This is the main difference between two-dimensional and three-dimension turbulence.

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Self-org. via Variational Principle

In the 2D Navier-Stokes turbulence, energy cascade inversely to low wave number and create large scale coherent structure. Let us consider the self-organized state in the HM model. A minimum enstrophy state with the constraint of fixed energy will be obtained by the variational principle,

$$\delta(W - \lambda E) = 0, \tag{21}$$

where λ is the Lagrange multiplier. The Euler-Lagrange equation is obtained to give,

$$\nabla^2\zeta + (1-\lambda)\zeta = 0, \tag{22}$$

$$\zeta = (1-\lambda)(n + \ln n_0). \tag{23}$$

If we assume $\partial_y = 0$, we obtain the solution as

$$\zeta = \zeta_0 \sin\sqrt{1-\lambda}x, \tag{24}$$

which suggests a zonal flow. The lowest eigenvalue gives the minimum enstrophy state, which has only one period.

Fluid Simulations

Turbulence driven zonal flows have been observed in the nonlinear simulations of various fluid turbulence model

- Hasegawa and Wakatani (1987)
  3D Simulation of HW model including magnetic curvature and shear in cylindrical geometry successefully generated shear
- Carreras *et al.* (1993)
  3D resistive interchange mode
  Electrostatic/Electromagnetic
  Background shear flow for symmetry-breaking and profile modification by turbulence
- Guzdar *et al.* (1993)
  3D drift resistive ballooning
- Sugama and Horton (1994)
  3D resistive interchange
  Comparison between their low dimensional dynamical model

Numerical Setup

The numerical code solves Hasegawa-Wakatani equation given by,

$$\frac{\partial \zeta}{\partial t} + \{\varphi, \zeta\} = \alpha(\varphi - n) + D_\zeta \nabla^2 \zeta \quad (25)$$

$$\frac{\partial n}{\partial t} + \{\varphi, n\} = \alpha(\varphi - n) - \kappa \frac{\partial \varphi}{\partial y} + D_n \nabla^2 n, \quad (26)$$

in the slab geometry of size $(2L)^2 = (2\pi/\Delta k)^2$ with $\Delta k = 0.15$. Small dissipations (viscosity D_ζ and diffusion coefficient D_n) are added to assure numerical stability.

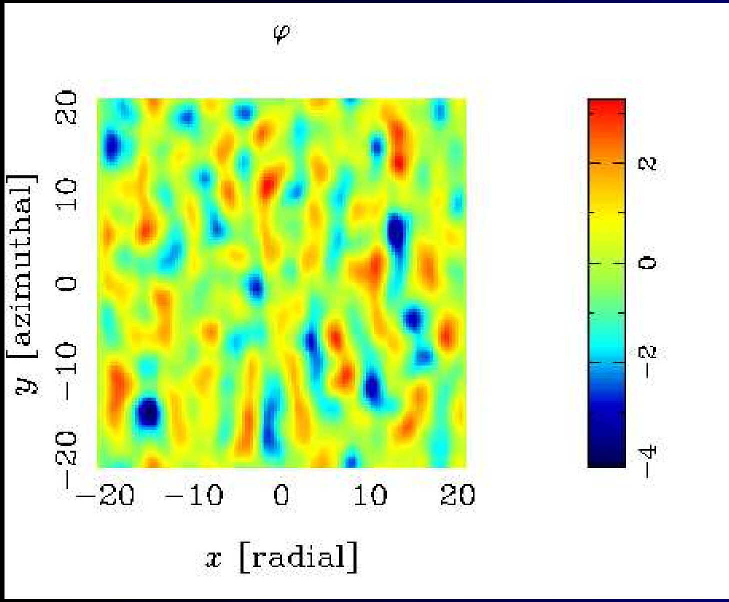
Periodic boundary in y direction and periodic or Dirichlet boundary in x direction,

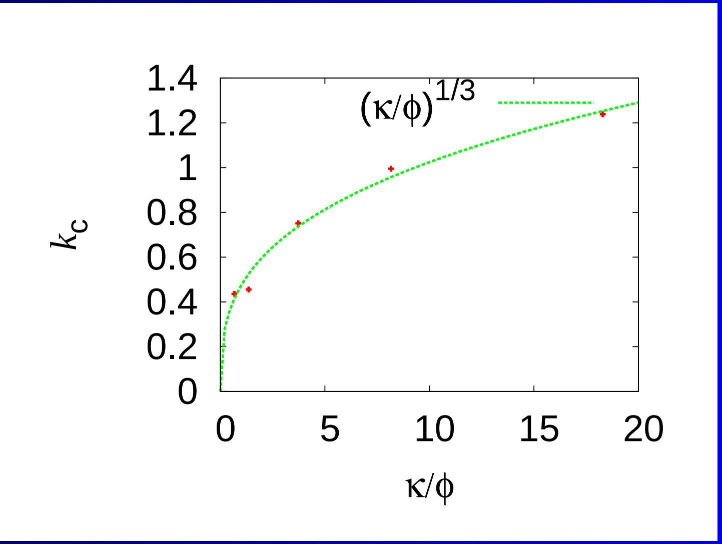
$$\varphi(\pm L) = 0, \;\; n(\pm L) = 0, \;\; \zeta(\pm L) = 0. \quad (27)$$

Time stepping algorithm is a 3rd order explicit linear multistep method.

Finite difference method is used for spatial discretization. Evaluation of the Poisson bracket term is performed by the Arakawa's method (Arakawa (1966)), which has third order acurracy and conserves energy and enstrophy.

Self-organized State in HM Model



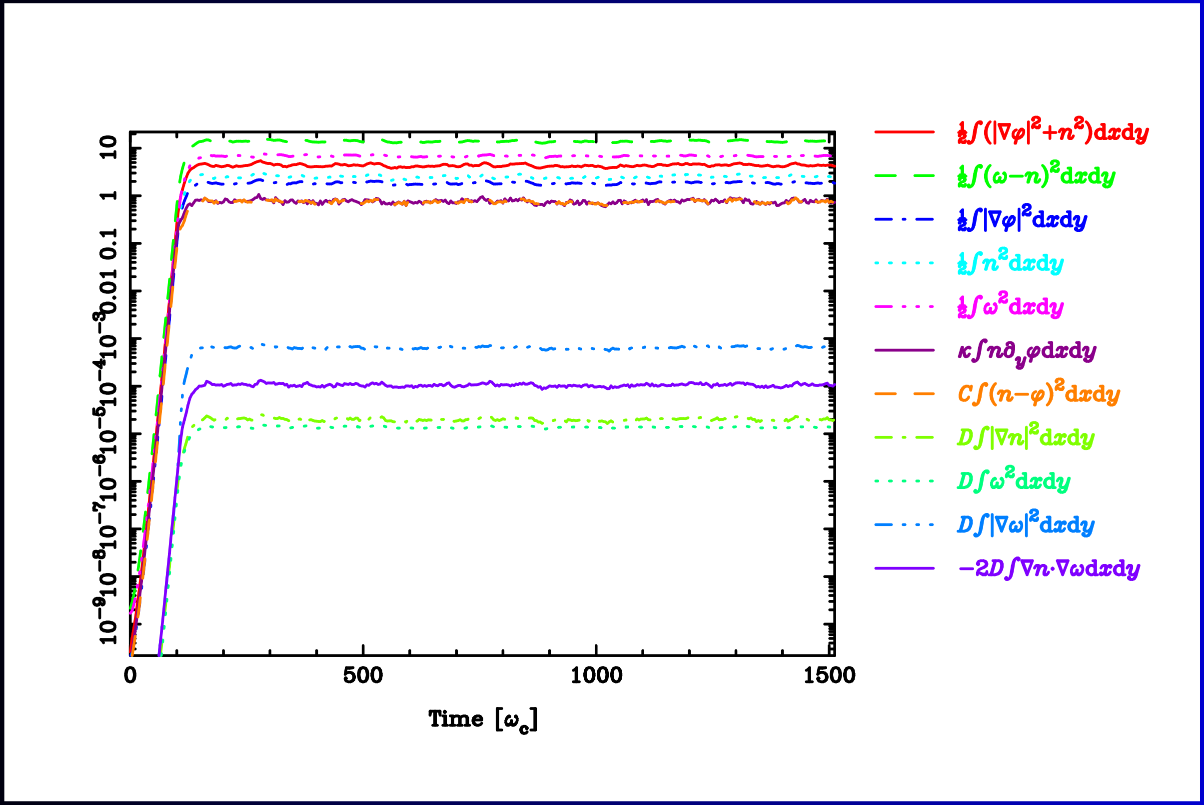


Left panel shows the final state of the stream function. Azimuthally (y) elongated structure can be seen although k_y is not zero.
Radial (x) mode number is determined to balance the nonlinear and the linear terms which demands $k_c \simeq (k/\varphi)^{1/3}$.

Rhines (1975); A. Hasegawa *et al.* (1979)

Linear Growth and Nonlin. Saturation

The figure shows the typical behavior of the simulation. Initial small perturbations linearly grow by the drift wave instability until nonlinearity comes into play a role. Nonlinear saturation occurs at the level where the drift wave drive and the resistive dissipation balance.



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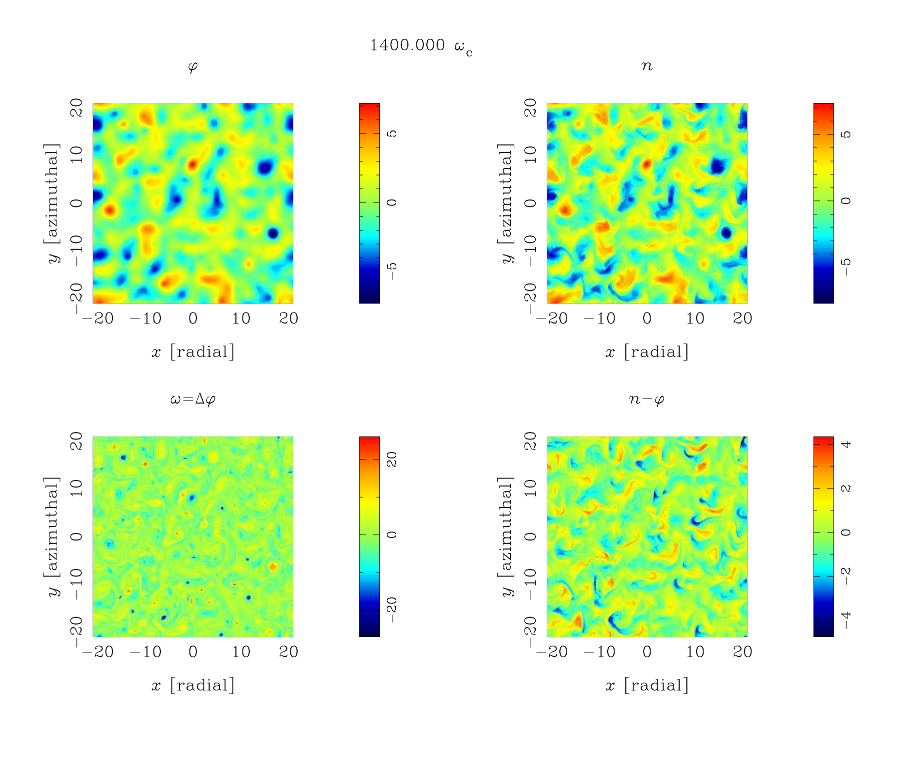
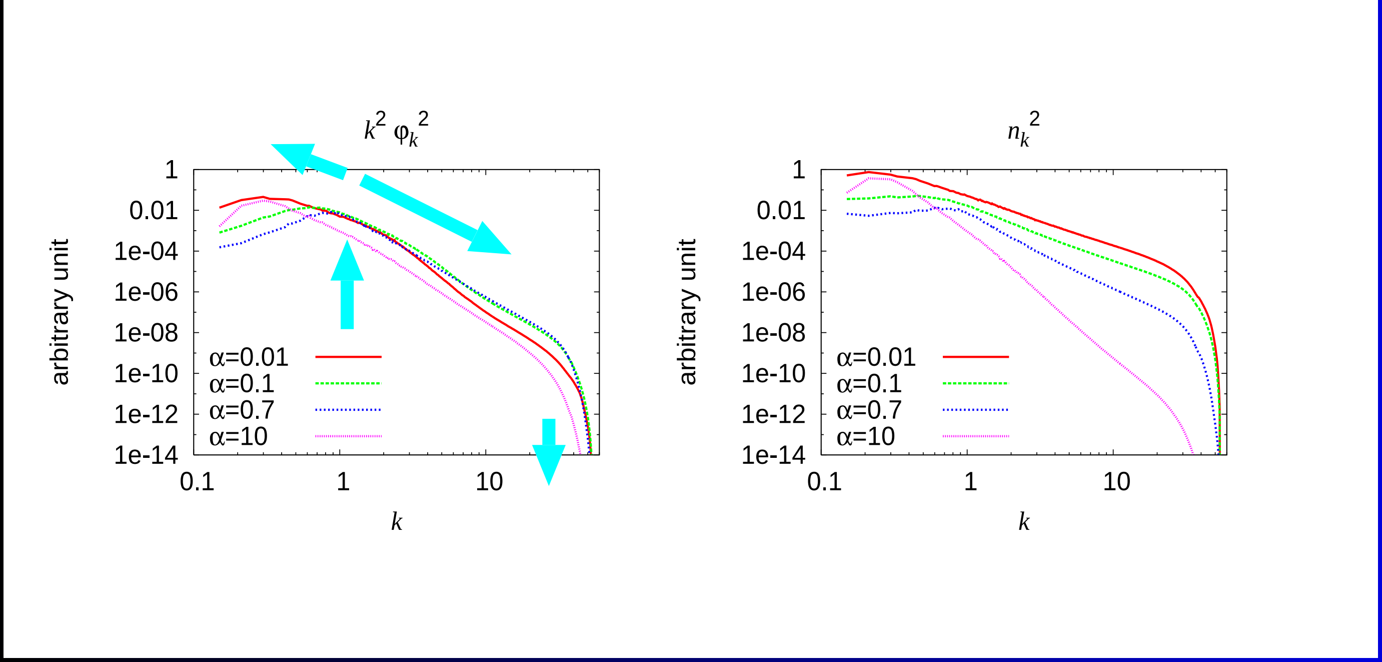
Saturated State (Periodic B.C.)



No anisotropic structure (zonal flow) is observed.

Energy and Enstrophy Spectra



- Intermediate cases ($\alpha = 0.1, 0.7$) do not show inverse energy cascade.
- Density spectrum depend on α

Power law indices [Camargo *et al.* (1995)]

α	E_k	E_k^V	E_k^n	U_k
0.1	-2.0	-2.9	-1.8	-1.6
0.7	-3.0	-3.2	-2.8	-1.7
10	-3.9	-3.6	-5.4	-2.2

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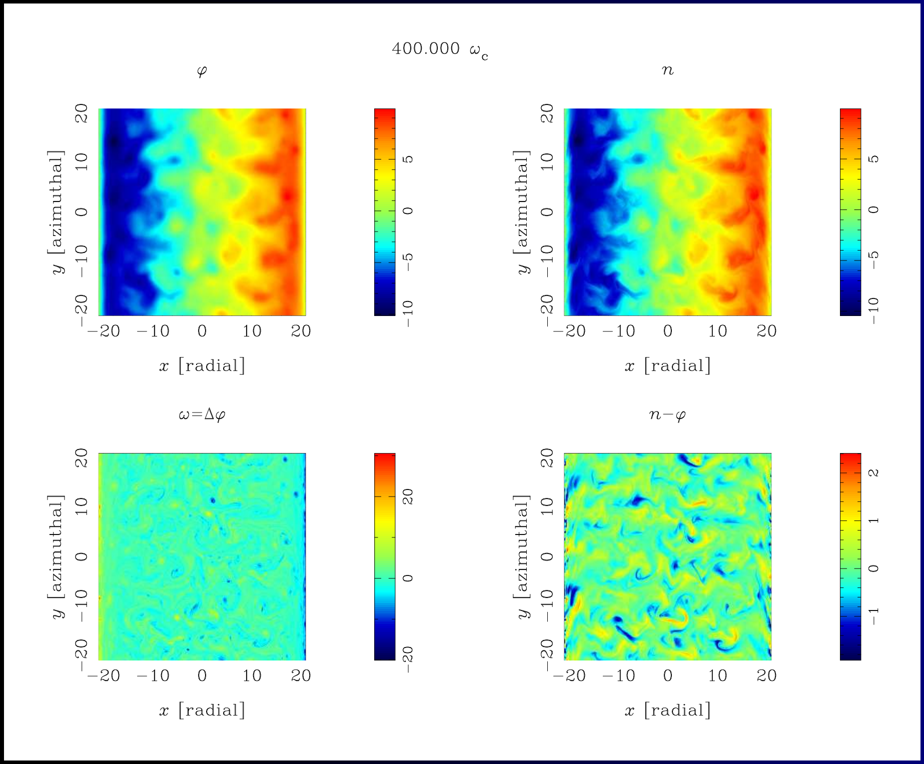
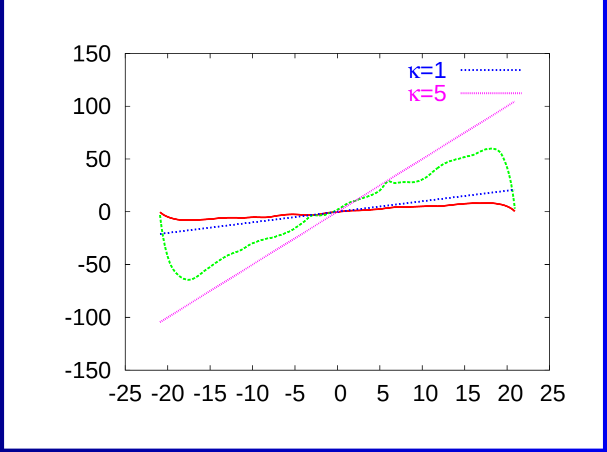
Saturated State (Fixed B.C.)





Background density profile is screened and turbulent drive is quenched

Modification of HW and HM Model

Because the zonal modes ($k_y = k_z = 0$) do not contribute to the parallel current, the resistive coupling term should have the form,

$$\alpha(\tilde{\varphi} - \tilde{n}) \tag{28}$$

where we defined the non-zonal component $\tilde{f} = f - \langle f \rangle$, and the zonal parts $\langle f \rangle \equiv 1/L \int_0^L f \mathrm{d}y$ (L is the periodic length in y).

The zonal components evolve through

$$\frac{\partial}{\partial t}\langle f \rangle - \frac{\partial}{\partial x}\left\langle f \frac{\partial \varphi}{\partial y} \right\rangle = 0. \tag{29}$$

where f stands for n and ζ.

In the HM limit ($\alpha \to \infty$), the non-zonal component of the density responds to the non-zonal potential, and the zonal density vanishes,

$$n = \tilde{n} = \tilde{\varphi}, \quad \langle n \rangle = 0. \tag{30}$$

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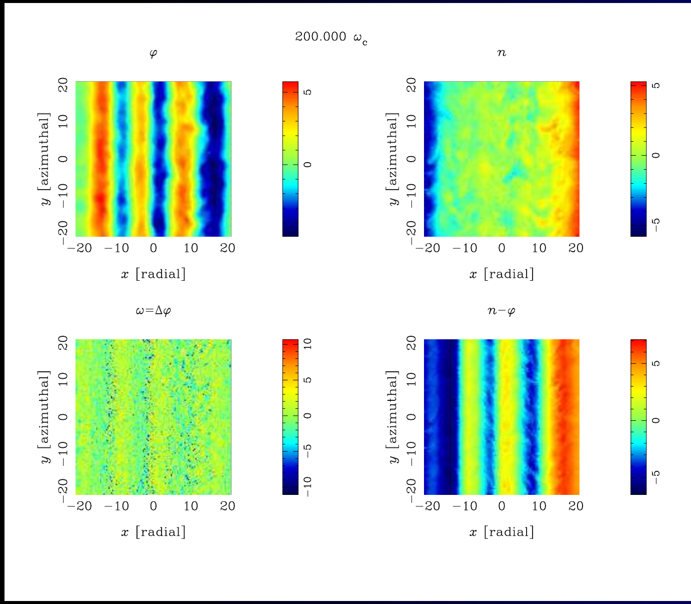
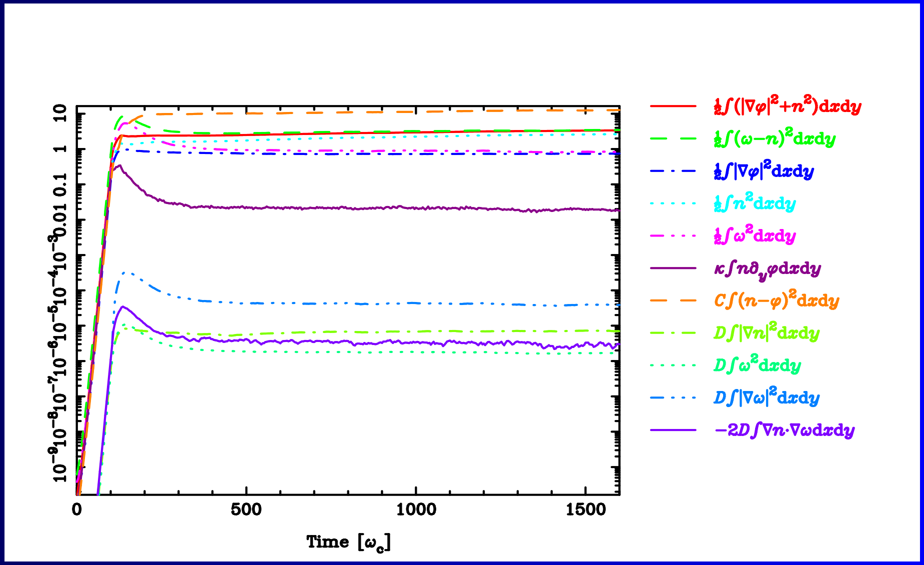
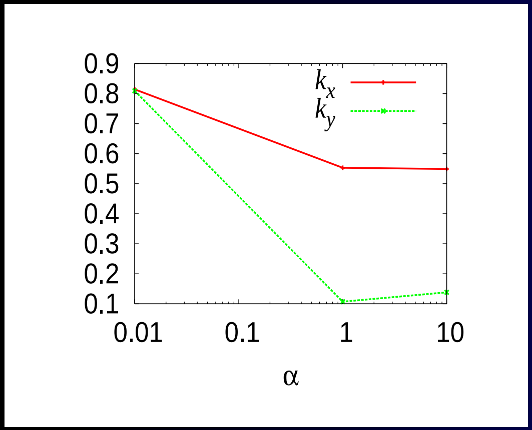
Zonal Flow in MHW Model







- Zonal flow is successfully generated MHW model.
- Zonal flow suppresses radial transport.
- Zonal flow does not exists in the hydro-dynamic case.

Summary and Conclusion

An interplay between direct numerical simulation and low dimensional dynamical model will provide us deeper insight of the physics of the L-H transition.

Key ingredient is an interaction between large scale zonal flow and turbulent fluctuations.

We can produce a zonal flow by a simple electrostatic 2-dimensional slab resistive drift wave model.

Does Hasegawa-Wakatani model have enough physics?

- electrostatic – electomagnetic
- 2 dimensional – 3 dimensional
- slab – cylindrical
- large aspect ratio limit – toroidal curvature
- background velocity shear
- magnetic shear
- nonlinear viscosity