Two-Fluid Nonlinear Simulation of Self-Organization of Plasmas with Flows

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Abstract

A three-dimensional Hall-Magnetohydrodynamic (Hall-MHD) simulation code has been developed to study the self-organization process in a two-fluid plasma. An appreciable amount of flow with a component perpendicular to the magnetic field is created in the two-fluid plasma. An adjustment of the ion helicity, which includes the Hall term, characterizes the two-fluid relaxation process.

Keywords: Two-fluid MHD, Self-organization, Double Beltrami equilibrium, Plasma flow, Nonlinear 3D simulation

1.Introduction

The Hall effect in a two-fluid plasma is represented by a singular perturbation term scaled by the ion skin depth, bringing about far richer structures created by flow-field coupling. Recent theory[1] predicts creation of a double Beltrami (DB) field that is in marked contrast with the Taylor relaxed state of the single-fluid MHD; the former may have a variety of flows.

In this paper, we study the dynamical process of the self-organization in two-fluid plasmas. We have developed a nonlinear three-dimensional (3D) simulation code. Starting from an unstable initial condition, through violent dynamics, total fields are rearranged, and new structure is reorganized. We can characterize such transient processes by some macroscopic quantities, that are "ideal constants of motion", i.e. energy and helicities. The ideal constants of motion conserve in the ideal limit. Dissipation, however, allows "adjustment" in these quantities. The adjustment of the ion helicity may play a role to create flows perpendicular to the magnetic field[2].

In quasi-static turbulence, perpendicular flows are described by a statistical average of fluctuations (Reynolds stress)[3]. Similar for generation of magnetic field, fluctuations can yield dynamo effects. A lot of studies related to those processes have been done in the single-fluid model. However, how the Hall effect alters those processes is still undeveloped (in recent studies[4], the Hall effects for dynamo processes have been addressed). In the self-organization process where drastic change of global structure occurs, we cannot define the statistical mechanical ensemble of fluctuations.

2.Theoretical background

2.1 Double Beltrami equilibrium

In an incompressible two-fluid model, the macroscopic evolution equations of a plasma can be cast into coupled vortex equations,

$$\frac{\partial}{\partial t}\boldsymbol{\omega}_j - \nabla \times (\boldsymbol{U}_j \times \boldsymbol{\omega}_j) = 0 \quad (j = 1, 2), \tag{1}$$

where $j = 1, 2$ indicate an electron and an ion. A pair of generalized vorticities and the corresponding flows are defined by

$$\begin{array}{rclrcl} \boldsymbol{\omega}_1 & = & \boldsymbol{B}, & \boldsymbol{\omega}_2 & = & \boldsymbol{B} + \epsilon\nabla \times \boldsymbol{V}, \\ \boldsymbol{U}_1 & = & \boldsymbol{V} - \epsilon\nabla \times \boldsymbol{B}, & \boldsymbol{U}_2 & = & \boldsymbol{V}, \end{array} \tag{2}$$

where $\boldsymbol{B}$ is the magnetic field, $\boldsymbol{V}$ is the ion flow velocity, and $\epsilon \equiv \delta_{\mathrm{i}}/L$ (L: system size, $\delta_{\mathrm{i}} \equiv c/\omega_{\mathrm{pi}}$: ion collisionless skin depth, c: speed of light, ω_{pi}: ion plasma frequency). The simplest equilibrium solution to (1) is given under the "Beltrami condition", that demands alignment of the vorticities along the corresponding flows;

$$\boldsymbol{B} = a(\boldsymbol{V} - \epsilon\nabla \times \boldsymbol{B}), \quad \boldsymbol{B} + \epsilon\nabla \times \boldsymbol{V} = b\boldsymbol{V}, \tag{3}$$

where a, b are constants. The general solution to (3) is given by a linear combination of two Beltrami fields,

$$\boldsymbol{B} = C_+\boldsymbol{G}_+ + C_-\boldsymbol{G}_-, \quad \boldsymbol{V} = (a^{-1} + \epsilon\lambda_+)C_+\boldsymbol{G}_+ + (a^{-1} + \epsilon\lambda_-)C_-\boldsymbol{G}_-, \tag{4}$$

where $\boldsymbol{G}_\pm$ are the Beltrami functions satisfying $\nabla \times \boldsymbol{G}_\pm = \lambda_\pm\boldsymbol{G}_\pm$, and $C_\pm$ are constants. The parameters $\lambda_\pm$, which are the eigenvalues of the curl operators, characterize the spatial scales of the vortices $\boldsymbol{G}_\pm$.

In the vortex dynamics equations (1), the general steady states are given by

$$\boldsymbol{U}_j \times \boldsymbol{\omega}_j = \nabla\varphi_j \quad (j = 1, 2), \tag{5}$$

where φ_j is a certain scalar field corresponding to the energy density. The Beltrami condition (3) gives a special class of solution such that both side of (5) equal to zero. The "generalized Bernoulli condition demands that the energy density is uniform in space.

From the original electron and ion momentum equations, we find that the generalized Bernoulli condition is given by the relation,

$$\frac{V^2}{2} + p = \text{const.} \tag{6}$$

2.2 Variational principle

The self-organization process of a plasma into relaxed states may be discussed by a variational principle. Taylor[5] assumed the minimization of the magnetic energy under the constraint of the magnetic helicity, and obtained the force-free state. This model is based on the "selective decay" implying that one of the constants of motion of the ideal limit decays faster than the others.

The variational principle for the two-fluid MHD requires a more generalized and rigorous arguments[2]. There are three global ideal invariants:

$$E = \frac{1}{2}\int (B^2 + V^2)\mathrm{d}x, \tag{7}$$

$$H_1 = \int \boldsymbol{A}\cdot\boldsymbol{B}\mathrm{d}x, \tag{8}$$

$$H_2 = \int (\boldsymbol{A} + \epsilon\boldsymbol{V})\cdot(\boldsymbol{B} + \epsilon\nabla\times\boldsymbol{V})\mathrm{d}x, \tag{9}$$

representing the total energy, the electron helicity, and the ion helicity. The minimization of a generalized enstrophy (measure of the complexity),

$$F = \frac{1}{2}\int |\nabla\times(\boldsymbol{A} + \epsilon\boldsymbol{V})|^2\mathrm{d}x, \tag{10}$$

with keeping E, H_1, and H_2 constant is carried out through the variation

$$\delta(F - \mu_0 E - \mu_1 H_1 - \mu_2 H_2) = 0, \tag{11}$$

where μ_0, μ_1, μ_2 are Lagrange multipliers. The general solution to the Euler-Lagrange equation is given by a linear combination of three Beltrami functions, which is not an equilibrium in general. The adjustment between three ideal constants of motion,

$$\delta(E - \mu_1' H_1 - \mu_2' H_2) = 0, \tag{12}$$

where μ_1', μ_2' are Lagrange multipliers, leads the DB field. The relaxation process is realized by minimizing perturbations, which is scaled by the generalized enstrophy F, with appropriate adjustments of macroscopic variables E, H_1, H_2.

3.Simulation model

To test the two-fluid self-organization in a more general dynamical framework, we consider a compressible, dissipative Hall-MHD plasma, governed by

$$\frac{\partial n}{\partial t} = -\nabla \cdot (n\boldsymbol{V}), \tag{13}$$

$$\frac{\partial (n\boldsymbol{V})}{\partial t} = -\nabla \cdot (n\boldsymbol{V}\boldsymbol{V}) - \nabla p + (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \frac{1}{R_{\mathrm{e}}}(\nabla^2 \boldsymbol{V} + \frac{1}{3}\nabla(\nabla \cdot \boldsymbol{V})), \tag{14}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times \left[\boldsymbol{V} \times \boldsymbol{B} - \frac{\epsilon}{n}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}\right] + \frac{1}{R_{\mathrm{m}}}\nabla^2 \boldsymbol{B}, \tag{15}$$

$$\frac{\partial p}{\partial t} = -(\boldsymbol{V} \cdot \nabla)p - \gamma p \nabla \cdot \boldsymbol{V} + \frac{1}{R_{\mathrm{e}}}|\nabla \times \boldsymbol{V}|^2 + \frac{4}{3R_{\mathrm{e}}}|\nabla \cdot \boldsymbol{V}|^2 + \frac{1}{R_{\mathrm{m}}}|\nabla \times \boldsymbol{B}|^2, \tag{16}$$

where n is the number density, p is the pressure, R_{e}, R_{m} are the Reynolds number and the Magnetic Reynolds number, and γ is the ratio of the specific heat. We have normalized variables by the scale length L and an appropriate measure of the magnetic field B_0 and the density n_0 (the ion mass m_{i} is assumed to be unity) as $\boldsymbol{x} \to L\boldsymbol{x}, \boldsymbol{B} \to B_0\boldsymbol{B}, n \to n_0 n, \boldsymbol{V} \to V_{\mathrm{A}}\boldsymbol{V}, t \to \tau_{\mathrm{A}} t, p \to (B_0^2/\mu_0)p$, where $V_{\mathrm{A}} \equiv B_0/\sqrt{\mu_0 m_{\mathrm{i}} n_0}$ is the Alfvén velocity, $\tau_{\mathrm{A}} \equiv L/V_{\mathrm{A}}$. The quasi-neutrality $n = n_{\mathrm{i}} = n_{\mathrm{e}}$, and zero electron pressure are assumed. The equations are solved by the finite difference and the Runge-Kutta-Gill methods.

The simulation domain is a rectangular box with size $2a \times 2a \times 2\pi R$, surrounded by a rigid perfect conducting wall. The system is periodic along the z axis. The boundary conditions in x, y directions are

$$\boldsymbol{n} \cdot \boldsymbol{B} = 0, \quad \boldsymbol{n} \times (\nabla \times \boldsymbol{B}) = 0, \quad \boldsymbol{V} = 0 \ \text{ at } \ x, y = \pm a. \tag{17}$$

To assure tangential components of the electric field vanish, we set $\epsilon = 0$ at the wall.

In order to confirm the validity of the simulation code, we have solved dispersion relations of purely transversal wave in a periodic domain. In an ideal limit, the dispersion relation for small perturbations around a uniform equilibrium, ($\boldsymbol{B}_0 = (0, 0, B_0)$, $\boldsymbol{V}_0 = 0$, n_0, p_0 are constants), is given by

$$\omega = \frac{V_{\mathrm{A}}}{2}\left[\pm\epsilon k^2 \pm \sqrt{\epsilon^2 k^4 + 4k^2}\right], \tag{18}$$

which describes the Alfvén whistler wave. In the limit $\epsilon \to 0$, we recover the dispersion relation of the shear Alfvén wave. Figure 1 shows the dispersion relation (18) for $\epsilon = 0,\ 0.1,\ 0.05$. The Hall term yields a dispersion effect to split the Alfvén wave in the electron and ion branches. The numerical results show good agreements with analytical curves.

4.Nonlinear simulations

The initial condition for nonlinear simulation was a 2D force free equilibrium that is also an equilibrium of the single-fluid model[6];

$$
\begin{aligned}
B_x &= -\frac{1}{k_0}(k_1 B_1 \cos k_2 x \sin k_1 y + k_2 B_2 \cos k_1 x \sin k_2 y), && (19)\\
B_y &= \frac{1}{k_0}(k_2 B_1 \sin k_2 x \cos k_1 y + k_1 B_2 \sin k_1 x \cos k_2 y), && (20)\\
B_z &= B_1 \cos k_2 x \cos k_1 y + B_2 \cos k_1 x \cos k_2 y, && (21)
\end{aligned}
$$

where B_1, B_2 are the amplitudes of two kinds of Fourier modes, $k_1 = n_1\pi/(2a)$, $k_2 = n_2\pi/(2a)$, $k_0 = \sqrt{k_1^2 + k_2^2}$, and n_1, n_2 are arbitrary integers. A uniform density $n_0 = 1$, and uniform pressure are assumed. The amplitude of the uniform pressure is given by a parameter $\beta = \int p \mathrm{d}x / \int B^2 \mathrm{d}x$. The initial condition has a flow such that $\boldsymbol{V}_0 = (0, 0, M_{\mathrm{A}} B_z/\sqrt{n_0})$, where M_{A} is the Alfvén Mach number.

We carried out two simulation runs. The simulation domain is implemented on $129 \times 129 \times 256$ point grids. The parameters are $\beta = 3$, $M_{\mathrm{A}} = 0.5$, and (a) $\epsilon = 0.1$, (b) $\epsilon = 0$. Figure 2 shows isosurfaces of the toroidal magnetic field at an initial time and $t = 50\tau_{\mathrm{A}}, 75\tau_{\mathrm{A}}, 100\tau_{\mathrm{A}}$. Two twisted columns in different color represent the columns of $B_z = \pm 0.3$. Because the initial condition is unstable against the ideal kink mode, initially assigned perturbations grow exponentially and plasma becomes turbulent. The magnetic reconnection process leads rearrangement of plasma configuration in Alfvénic time scale. The plasma settled into a twisted state after the turbulent relaxation process, and it flows both in the toroidal and poloidal direction as time goes. Figure 3 shows the ratio of the perpendicular and the parallel components of the kinetic energy. The perpendicular component remains more than 10% for $\epsilon = 0.1$, while it does only few percent for $\epsilon = 0$. This result agrees with the theoretical prediction, and highlights the difference from the Taylor relaxation in the single-fluid MHD. The simulation result demonstrates creation of quasi-static stable state ($\partial_t \simeq 0$), where the energy density $(p + \frac{1}{2}V^2)$ is almost homogenized (fluctuation of the energy density is small, $|\nabla(p + \frac{1}{2}V^2)|/|\boldsymbol{j} \times \boldsymbol{B}| \simeq 0.1$). Hence, the homogeneity of the Beltrami parameters implies the creation of the DB field. In Fig. 4, we show the distribution of the Beltrami parameters a, b in the poloidal plane. The figures show that a, b are almost uniform in the magnetic columns.

We have investigated the variational principle in the Hall-MHD plasma. Figure 5 shows the time evolution of the macroscopic quantities: E, H_1, $H_2' \equiv H_2 - H_1$, and

$$F' = \frac{1}{2}\int |\nabla \times \boldsymbol{V}|^2 \mathrm{d}x, \tag{22}$$

normalized by the initial values for the case (a). The minimization of (10) or (22) will give the same result. The energy and two helicities monotonically decrease due to the dissipations. However, the dissipation rate of the ion helicity is faster than that of others. Since the ion helicity includes the highest order derivative term $(\boldsymbol{V} \cdot \nabla \times \boldsymbol{V})$, it is selectively adjusted during the relaxation process. The generalized enstrophy increases in the relaxation phase because it is not an ideal constants of motion. After the relaxation phase $t \simeq 20 \sim 40\tau_{\mathrm{A}}$, it decreases faster than other three quantities. The relaxed state is characterized not by minimizing the energy, but by minimizing the generalized enstrophy.

5.Summary

We have developed a Hall-MHD simulation code in a 3D rectangular domain, and have studied the self-organization process of a flow-field coupled state. Comparing the relaxed state in the two-fluid model with that in the single-fluid model, an appreciable flow with a component perpendicular to the magnetic field is created. The perpendicular flow highlights the distinct character of the two-fluid relaxed state (the DB state). The relaxation process have also been investigated by means of the time evolution of the macroscopic quantities. Since the ion helicity decays faster than the energy, the selective decay concepts of the energy cannot work in the two-fluid model. The enstrophy (a measure of fluctuations) minimization leads the relaxation. During the relaxation process, the ion helicity (the most fragile macroscopic quantity) is severely adjusted to obtain the DB state.

In the two-fluid relaxation, the ion helicity plays an essential role for the relaxation process. The ion helicity includes the Hall term, and its counterpart in the single-fluid model is the cross helicity $H_{\mathrm{C}} = \int \boldsymbol{V} \cdot \boldsymbol{B} \mathrm{d}x$. The relaxation model due to the variational principle by use of the cross helicity only leads the single Beltrami magnetic field (the Taylor state) and the parallel flow. Thus, we conclude that the Hall term generates the perpendicular flow in the two-fluid plasmas. Physically, the Hall current is induced by deviation of ion motions from electron motions. The Beltrami condition demands this Hall current to support the perpendicular flows.

References

[1] S.M. Mahajan and Z. Yoshida, Phys. Rev. Lett. **81**, 4863 (1998).
[2] Z. Yoshida and S.M. Mahajan, Phys. Rev. Lett. **84**, 095001 (2002).

[3] P.W. Terry *et al.*, Rev. Mod. Phys. **72**, 109 (2003).

[4] P.D. Mininni *et al.*, Astrophys. J. **587**, 472 (2003).

[5] J.B. Taylor, Phys. Rev. Lett. **33**, 1139 (1974).

[6] R. Horiuchi and T. Sato, Phys. Rev. Lett. **55**, 211 (1985).

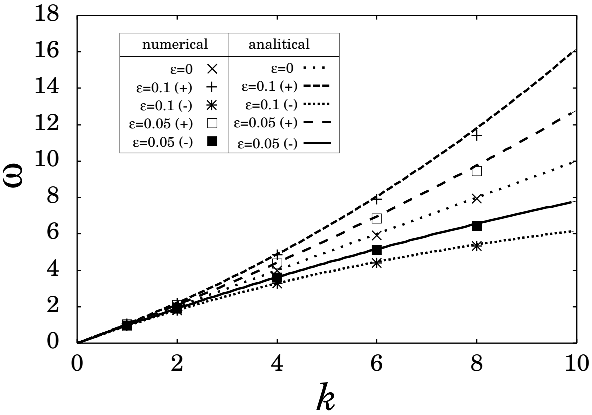


Figure 1: Dispersion relations of the Alfvén whistler wave.

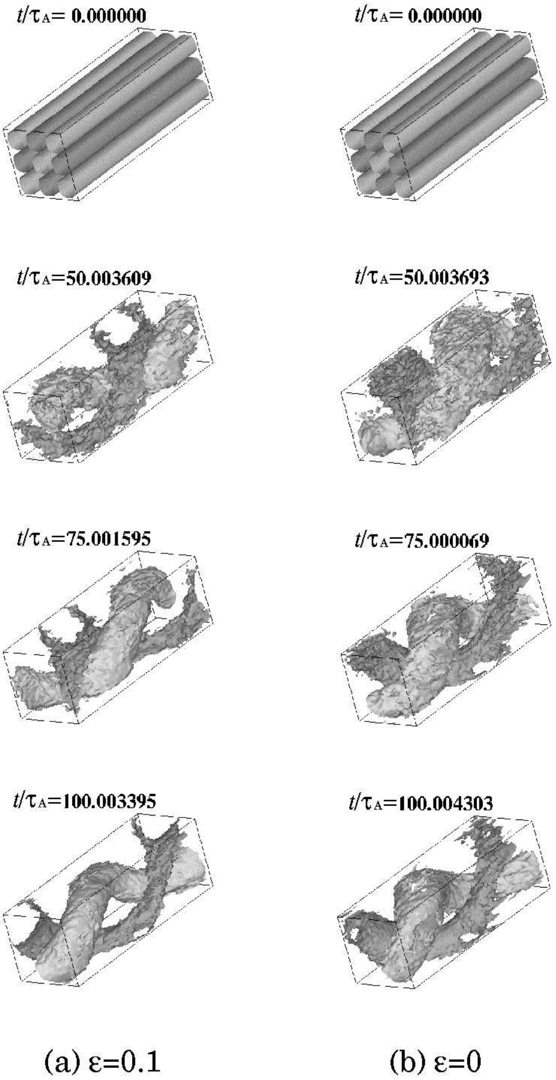


Figure 2: Isosurfaces of the toroidal magnetic field.

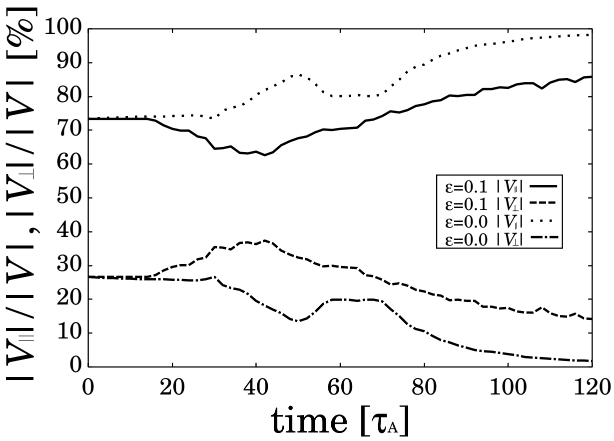
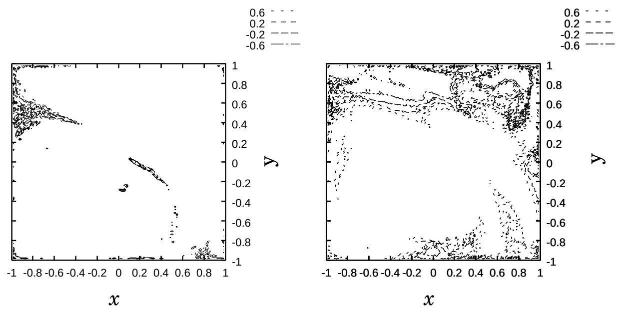


Figure 3: Ratio of components of the kinetic energy.



(a) distribution of a (b) distribution of b

Figure 4: Distribution of the Beltrami parameters a, b in the poloidal plane. The figures show that a, b are almost uniform in the magnetic columns.

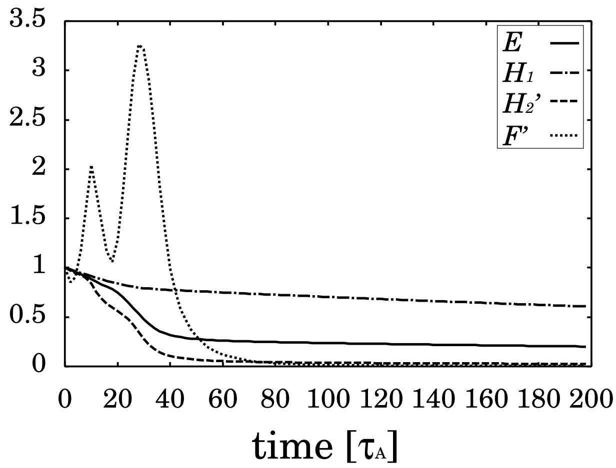


Figure 5: Time evolution of the macroscopic quantities. The enstrophy minimization with the selective adjustment of the ion helicity characterizes the relaxation in the two-fluid plasma.