Two-Fluid Nonlinear Simulation of Self-Organization of Plasmas with Flows

The 13th International Toki Conference
2003/12/12

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Introduction

Flows in Plasmas

- Laboratory Plasmas, Fusion Plasmas
  (Non-neutral Plasmas, Tokamak H-mode boundary layer, Shear-flow Stabilization)
- Space Plasmas
  (Magnetic reconnection, Solar corona, High beta equilibrium of the Jupiter)

Single-Fluid Model

- Singularity in equilibrium equation

\[ L = \int \left[ \frac{1}{2} (1 - M^2(\psi)) |\nabla \psi|^2 - F(\psi) \right] \, dx \Rightarrow -\Delta [\psi - \phi(\psi)] = F'(\psi) \]  

(1)


- Scale invariance
  Current sheet model to heating the solar corona
Two-Fluid MHD

The Hall term leads a nonlinear singular perturbation, which
- puts anchor on unlimited scale conversions induced by nonlinearity.
- avoids singularity.
⇒ cf. viscosity in neutral fluids

**Double Beltrami Equilibrium**

\[
\frac{\partial}{\partial t} \omega_j - \nabla \times (U_j \times \omega_j) = 0 \quad (j = 1, 2) \quad (2)
\]

\[
\begin{align*}
\omega_1 &= B, \\
\omega_2 &= B + \epsilon \nabla \times V \\
U_1 &= V - \epsilon \nabla \times B, \\
U_2 &= V
\end{align*} \quad (3)
\]

\[
\epsilon = \delta_i / L \quad (\delta_i \equiv c / \omega_{pi} \text{: ion collisionless skin depth})
\]

**Beltrami Bernoulli Condition**

\[
U_j \times \omega_j = 0 = \nabla \varphi_j \quad (j = 1, 2) \quad (4)
\]
Double Betrami Equilibrium

The general solution is described by a linear combination of eigenfunctions of the curl operator \( \mathbf{G}_\pm (\nabla \times \mathbf{G}_\pm = \lambda_\pm \mathbf{G}_\pm) \):

\[
\mathbf{B} = C_+ \mathbf{G}_+ + C_- \mathbf{G}_-, \quad \mathbf{V} = (a^{-1} + \epsilon \lambda+)C_+ \mathbf{G}_+ + (a^{-1} + \epsilon \lambda-)C_- \mathbf{G}_- \quad (5)
\]

The generalized Bernoulli condition translates as

\[
p + \frac{V^2}{2} = \text{const.} \quad (6)
\]

Application

- H-mode boundary layer
- Eruptive events in solar coronas
- Chaos-induced dissipation in collisionless magnetic reconnection
Relaxation Process

Global Invariants

\[
E = \frac{1}{2} \int (B^2 + V^2) \, dx \quad (7)
\]

\[
H_1 = \frac{1}{2} \int \mathbf{A} \cdot \mathbf{B} \, dx \quad (8)
\]

\[
H_2 = \frac{1}{2} \int (\mathbf{A} + \epsilon \mathbf{V}) \cdot (\mathbf{B} + \epsilon \nabla \times \mathbf{V}) \, dx \quad (9)
\]

Variational Principle


\[
\delta (F - \mu_0 E - \mu_1 H_1 - \mu_2 H_2) = 0 \quad (10)
\]

\[
F = \frac{1}{2} \int |\nabla \times (\mathbf{A} + \epsilon \mathbf{V})|^2 \, dx \quad (11)
\]

\[
\delta (E - \mu'_1 H_1 - \mu'_2 H_2) = 0 \quad (12)
\]

\(\mu_0, \mu_1, \mu_2\) are Lagrange multipliers. The relaxation process is realized by minimizing perturbation \(F\) with appropriate adjustment of \(E, H_1, H_2\).
Simulation Model

We consider compressible Hall-MHD equations

\[
\frac{\partial n}{\partial t} = -\nabla \cdot (n \mathbf{V}) \quad (13)
\]

\[
\frac{\partial (n \mathbf{V})}{\partial t} = -\nabla \cdot (n \mathbf{V} \mathbf{V}) - \nabla p + (\nabla \times \mathbf{B}) \times \mathbf{B} + \frac{1}{R_e} (\nabla^2 \mathbf{V} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{V})) \quad (14)
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left[ \mathbf{V} \times \mathbf{B} - \frac{\epsilon}{n} (\nabla \times \mathbf{B}) \times \mathbf{B} \right] + \frac{1}{R_m} \nabla^2 \mathbf{B} \quad (15)
\]

\[
\frac{\partial p}{\partial t} = - (\mathbf{V} \cdot \nabla) p - \gamma p \nabla \cdot \mathbf{V} + \frac{1}{R_e} |\nabla \times \mathbf{V}|^2 + \frac{4}{3R_e} (\nabla \cdot \mathbf{V})^2 + \frac{1}{R_m} |\nabla \times \mathbf{B}|^2 \quad (16)
\]

where \( n \): number density (ion mass is normalized to be unity), \( p \): ion pressure, \( R_e \): Reynolds number, \( R_m \): magnetic Reynolds number, \( \gamma \): ratio of specific heat. We have assumed the quasi-neutrality \( n = n_i = n_e \), and \( p_e = 0 \).
Simulation Model

Simulation Domain
3 dimensional rectangular domain \([-a, a] \times [-a, a] \times [0, 2\pi R]\)

Boundary Condition
Periodic in \(z\) direction
Rigid perfect conducting wall in \(x, y\) direction

\[
\begin{align*}
\nu &= 0 \\
B_N &= 0 \\
\partial_N B_T &= 0 \quad (n \times (\nabla \times B) = 0)
\end{align*}
\]  

where \(n\) is a unit normal vector, \(\partial_N\) is a normal derivative, and \(B_N, B_T\) is the normal and tangential component of the magnetic field, respectively.
To assure tangential components of the electric field vanish, we set \(\epsilon = 0\) at the wall.

Numerical Method
2nd order central difference in space, the Runge-Kutta-Gill method for time integration
2nd order smoothing to suppress short wave length mode
add a diffusion in the equation of continuity to keep density positive
In an ideal limit, the dispersion relation for purely transversal small perturbation around a uniform equilibrium \( \mathbf{B}_0 = (0, 0, B_0), \mathbf{V}_0 = 0, \rho_0, p_0 \) are constants) is given by

\[
\omega = \frac{V_A}{2} \left[ \pm \epsilon k^2 \pm \sqrt{\epsilon^2 k^4 + 4k^2} \right],
\]  (21)

which describes the Alfvén whistler wave. We have solved numerically this dispersion relation in a periodic domain in order to confirm a validity of the code.
Initial Condition

2D force free equilibrium ($\nabla \times B = k_0 B$)

Magnetic Field

\[
B_x = -\frac{1}{k_0} (k_1 B_1 \cos k_2 x \sin k_1 y + k_2 B_2 \cos k_1 x \sin k_2 y) \tag{22}
\]

\[
B_y = \frac{1}{k_0} (k_2 B_1 \sin k_2 x \cos k_1 y + k_1 B_2 \sin k_1 x \cos k_2 y) \tag{23}
\]

\[
B_z = B_1 \cos k_2 x \cos k_1 y + B_2 \cos k_1 x \cos k_2 y \tag{24}
\]

Flow

\[
\mathbf{V} = V_z \mathbf{e}_z = M A B_z \mathbf{e}_z \tag{25}
\]

Density

\[
\rho = 1 \ (\text{uniform}) \tag{26}
\]

Pressure

\[
p = p_0 \ (\text{uniform}) \tag{27}
\]

\[
\beta = \int p_0 \, dx / \int \frac{1}{2} B^2 \, dx \tag{28}
\]
Equilibrium State

Aspect Ratio: \(2\pi R/2a = 3\)
Grid: \(129 \times 129 \times 256\)
\(R_e = 10^4, R_m = 10^4\)
\(\beta = 3.0, M_A = 0.5 \epsilon = 0, 0.1\)

\(\text{left panel: } \epsilon = 0.1 \text{ (Hall-MHD)}, \text{ right panel: } \epsilon = 0 \text{ (MHD)}\)
**Kinetic Energy**

- $\epsilon = 0.1$ (Hall-MHD): $V_\perp$ is more than 10% of $|V|$.
- $\epsilon = 0.0$ (MHD): No $V_\perp$ remains.
Beltrami Conditions

\[ B = a(V - \frac{\epsilon}{n} \nabla \times B), \quad B + \epsilon \nabla \times V = bV \]  

Time : \(100.0 \tau_A\)

(a) distribution of \(a\)  
(b) distribution of \(b\)
Temporal evolution of the conservative quantities and the generalized enstrophy normalized by the initial value for the case of $\epsilon = 0.1$

The coercivity condition demands that the order of fragility is $H_1 > E > H_2 - H_1$, which agree with the numerical result.
Summary

- We have developed a nonlinear 3D Hall-MHD simulation code.
- The dispersion relation of the Alfvén whistler wave is reproduced.
- Comparing the two-fluid relaxed state with single-fluid one, an appreciable fbw including perpendicular component was created.
- Relaxation process is investigated by means of the variational principle. Energy is not a minimizer.