Equilibrium of kinetic equation

Ryusuke Numata
The University of Maryland
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1 Vlasov-Maxwell equations

The Vlasov and Maxwell equations are given by,

$$\frac{\partial f_s}{\partial t} + v \cdot \frac{\partial f_s}{\partial x} + \frac{q_s}{m_s} (E + v \times B) \cdot \frac{\partial f_s}{\partial v} = 0,$$

$$\nabla \times E = -\frac{\partial B}{\partial t},$$

$$\nabla \times B = \mu_0 \sum_s q_s \int f_s v \, dv + \frac{1}{c^2} \frac{\partial E}{\partial t},$$

$$\nabla \cdot E = \frac{1}{\epsilon_0} \sum_s q_s \int f_s \, dv,$$

$$\nabla \cdot B = 0.$$  

$f_s$ is the distribution function in a six dimensional phase space spanned by $(x, v)$. $q_s, m_s$ are the electric charge, the mass. The subscript $s$ denotes the species. $E, B$ are the electric and the magnetic fields, $\mu_0, \epsilon_0, c = 1/\sqrt{\mu_0 \epsilon_0}$ are the vacuum permeability, the vacuum permittivity, and the speed of light, respectively.

We rewrite the electromagnetic field in terms of the potentials $\phi, A$,

$$E = - \nabla \phi - \frac{\partial A}{\partial t},$$

$$B = \nabla \times A,$$

which automatically satisfy the Faraday’s induction equation (2) and the divergence free condition of $B$ (5). We ignore the displacement current (the second term in the right-hand side) in the Ampère’s law (3). By imposing the Coulomb gauge condition, $\nabla \cdot A = 0$, the potentials are governed by

$$-\nabla^2 A = \mu_0 \sum_s q_s n_s U_s = \mu_0 j,$$

$$-\nabla^2 \phi = \frac{1}{\epsilon_0} \sum_s q_s n_s.$$
We have defined the first and the second moments of $f$ as follows,

\[ n_s = \int f_s dv, \]  
\[ u_s = \frac{1}{n_{0s}} \int f_s uv dv, \]

where we call $n_s$ the number density, and $u_s$ the bulk flow. The current density is given by

\[ j = \sum_s q_s n_s u_s. \]

## 2 Harris sheet

We seek a thermal equilibrium solution to the Vlasov-Maxwell system depending only on $x$. The equations which the equilibrium solution should satisfy are

\[ \mathbf{v} \cdot \frac{\partial f_s}{\partial x} + \frac{q_s}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0, \]  
\[ \nabla^2 \mathbf{A} = \mu_0 \sum_s q_s \int f_s u dv \]  
\[ \nabla^2 \phi = \frac{1}{\epsilon_0} \sum_s q_s \int f_s dv. \]

The electromagnetic fields are given by

\[ \mathbf{E} = -\nabla \phi = \left( \begin{array}{c} -\frac{d\phi}{dx} \\ 0 \\ 0 \end{array} \right), \quad \mathbf{B} = \nabla \times \mathbf{A} = \left( \begin{array}{c} 0 \\ -\frac{dA_z}{dx} \\ 0 \end{array} \right). \]

Particle dynamics is described solely by the Hamiltonian

\[ H = \frac{(P - qA(x))^2}{2m} + q\phi(x). \]

Since we are considering the one dimensional steady state, the Hamiltonian does not depend on $t$, $y$, and $z$, then, there exists three constants of motion,

\[ P_y = mv_y \]
\[ P_z = mv_z + qA_z \]
\[ H = \frac{P_x^2 + P_y^2 + (P_z - qA_z)^2}{2m} + q\phi = \frac{m}{2}(v_x^2 + v_y^2 + v_z^2) + q\phi. \]

Due to those constants of motion, the particle dynamics is integrable, and, the distribution function of the form,

\[ f = f(P_y, P_z, H), \]
gives the complete solution to the Vlasov equation \(^1\).

We next find the thermal equilibrium solution by minimizing the (Shannon’s information) entropy,

\[ S = \int f \ln f \, dv, \]  

while keeping the total energy, the total momenta, and the number density,

\[ E = \int H f \, dv, \]  
\[ M_y = \int P_y f \, dv, \]  
\[ M_z = \int P_z f \, dv, \]  
\[ n = \int f \, dv, \]  

constants. The variation

\[ \delta(S + \lambda_1 E + \lambda_2 M_y + \lambda_3 M_z + \lambda_4 n) = 0 \]  

leads to the Euler-Lagrange (E-L) equation,

\[ (1 + \lambda_4) + \lambda_1 H + \lambda_2 P_y + \lambda_3 P_z + \ln f = 0 \]  

where \(\lambda_{1,2,3,4}\) are the Lagrange multiplier. The solution to the E-L equation is given by

\[ f(P_y, P_z, H) = A_1 \exp \left[ -\frac{\lambda_1 H}{T} - \frac{\lambda_2 P_y}{T} - \frac{\lambda_3 P_z}{T} \right]. \]  

It is more physically intuitive to write

\[ \lambda_1 = \frac{1}{T}, \quad \lambda_2 = -\frac{u_y}{T}, \quad \lambda_3 = -\frac{u_z}{T}, \]  

where \(T\) is the temperature.

Then, the distribution function is written as

\[ f = A_2 \exp \left[ -\frac{(v - u_v)}{v_{th}} + \frac{q(u \cdot A - \phi)}{T} \right], \]  

\(^1\)We show the distribution function satisfies the Vlasov equation. Obviously, the time derivative vanishes,

\[ \frac{\partial f}{\partial t} = \frac{dP_y}{dt} \frac{\partial f}{\partial P_y} + \frac{dP_z}{dt} \frac{\partial f}{\partial P_z} + \frac{dH}{dt} \frac{\partial f}{\partial H} = 0. \]  

The Vlasov equation is rewritten as

\[ v_x \frac{\partial f}{\partial x} + \frac{q}{m} \left( \frac{\partial \phi}{\partial x} + v_y \frac{dA_y}{dx} + v_z \frac{dA_z}{dx} \right) \frac{\partial f}{\partial P_z} - v_y \frac{dA_y}{dx} \frac{\partial f}{\partial P_z} = 0. \]  

We then rewrite the derivative of \(f\) in terms of the constants of motion.

\[ \frac{\partial f}{\partial x} = \frac{\partial f}{\partial P_x} \frac{\partial P_x}{\partial x} + \frac{\partial f}{\partial H} \frac{\partial H}{\partial x} = q \frac{\partial A_x}{dx} \frac{\partial f}{\partial P_x} + q \frac{\partial \phi}{dx} \frac{\partial f}{\partial H}, \]  
\[ \frac{\partial f}{\partial v_x} = \frac{\partial f}{\partial H} \frac{\partial H}{\partial v_x} = mv_x \frac{\partial f}{\partial H}, \]  
\[ \frac{\partial f}{\partial v_y} = \frac{\partial f}{\partial P_y} \frac{\partial P_y}{\partial v_y} + \frac{\partial f}{\partial H} \frac{\partial H}{\partial v_y} = m \frac{\partial f}{\partial P_y} + mv_y \frac{\partial f}{\partial H}. \]  

Plugging all into the Vlasov equation, we obtain

\[ v_x \left( q \frac{dA_x}{dx} \frac{\partial f}{\partial P_x} + q \frac{\partial \phi}{dx} \frac{\partial f}{\partial H} \right) + \frac{q}{m} \left( \frac{\partial \phi}{dx} + v_x \frac{dA_x}{dx} \right) mv_x \frac{\partial f}{\partial H} - \frac{q}{m} \left[ v_x \frac{dA_x}{dx} \left( m \frac{\partial f}{\partial P_x} + mv_x \frac{\partial f}{\partial H} \right) \right] = 0. \]
where $A_2 = A_1 \exp(u^2/v_{th}^2)$ (note that $u$ is a constant), $v_{th} = \sqrt{2T/m}$. By normalizing $f$, we get

$$f = \frac{n}{\pi^{3/2} v_{th}} \exp \left[ - \left( \frac{v - u}{v_{th}} \right)^2 \right] \exp \left[ \frac{q(u \cdot A - \phi)}{T} \right]. \quad (37)$$

We now consider the two species plasma consists of electrons ($q_e = -e$) and hydrogen ions ($q_i = e$), and find the consistent field, having the form $\phi = 0$, $A = A_z(x)$. The equations to be satisfied are the quasi-neutrality (came from the Poisson’s equation (15)), and the Ampère’s law in the $z$ direction,

$$\int f_i dv - \int f_e dv = 0, \quad (38)$$
$$- \frac{d^2 A_z}{dx^2} = \mu_0 \left( q_i \int f_i v_z dv + q_e \int f_e v_z dv \right). \quad (39)$$

To have a finite $A_z$ and $A_y = 0$, we take $u = u_z z$. The quasi neutrality condition demands

$$n_0 e \exp \left[ \frac{q_e u_z A_z}{T} \right] - n_0 i \exp \left[ \frac{q_i u_z A_z}{T_i} \right] = 0. \quad (40)$$

If we assume $q_i u_z/T_i = q_e u_{ez}/T_e$, the condition is reduced to $n_0 i = n_0 e = n_0$. The first order moment in the $z$ direction reads

$$M^{(1)}_z = n_0 u_z \exp \left[ \frac{q u_z A_z}{T} \right]. \quad (41)$$

Then, the Ampère’s law yields

$$- \frac{d^2 A_z}{dx^2} = \mu_0 e n_0 (u_{iz} - u_{ez}) \exp \left[ \frac{q u_z A_z}{T} \right] = G \exp(\kappa A_z), \quad (42)$$

where

$$G = \mu_0 e n_0 (u_{iz} - u_{ez}), \quad \kappa = q u_z / T. \quad (43)$$

The general solution is

$$A_z(x) = C_1 \ln \cosh(x/a) + A_{z,h}. \quad (44)$$

$A_{z,h}$ is the homogeneous part, $d^2 A_{z,h}/dx^2 = 0$, which we don’t care about (we set $A_{z,h} = 0$). The derivative of $A_z$ gives

$$\frac{dA_z}{dx} = \frac{C_1}{a} \tanh(x/a), \quad (45)$$
$$\frac{d^2 A_z}{dx^2} = \frac{C_1}{a^2} \frac{1}{\cosh^2(x/a)}. \quad (46)$$

Noting that $\exp(\kappa A_z) = \cosh^{\kappa C_1}(x/a)$, to satisfy the Ampère’s law, we demand $C_1 = -2/\kappa = -2T/(qu_z)$. Another relations to be satisfied is

$$- \frac{C_1}{a^2} = G = \mu_0 e n_0 (u_{iz} - u_{ez}) = - \mu_0 e n_0 \left( 1 + \frac{T_i}{T_e} \right) u_{ez}. \quad (47)$$
which gives the current sheet width $a$,
\[
\frac{1}{a} = \frac{1}{d_a} \left( \frac{u_{iz}}{v_{thi}} \right) \left( 1 + \frac{T_i}{T_c} \right)^{1/2} = \frac{1}{d_e} \left( \frac{u_{ez}}{v_{the}} \right) \left( 1 + \frac{T_i}{T_c} \right)^{1/2}.
\] (48)

$d_a$ is the inertial skin depth. Note that $d_a v_{th} = \lambda_D c$ where $\lambda_D$ is the Debye length. Finally, the vector potential is given by
\[
A_z(x) = -\frac{2T}{qu_z} \ln \cosh(x/a).
\] (49)

$T/(qu_z)$ does not depend on the species. The magnetic field and the density now become
\[
B_y(x) = \frac{\sqrt{2\mu_0 n_0(T_i + T_e) \tanh(x/a)}}{n_0} = B_0 \tanh(x/a),
\] (50)
\[
n(x) = n_0 \cosh^{-2}(x/a),
\] (51)
where $B_0^2/2\mu_0 = n_0(T_i + T_e)$, or $\beta = 2\mu_0 n_0(T_i + T_e)/B_0^2 = 1$. The relation $C_1/a = B_0$ gives the diamagnetic drift velocity, $u_{sz} = -2T/(qu_0 a)$.

We check this solution to the kinetic equation also satisfies the steady fluid equation. We write the steady two fluid equations,
\[
\nabla \cdot (n_s u_s) = 0
\] (52)
\[
n_s m_s (u_s \cdot \nabla) u_s = -\nabla p_s + n_s q_s (E + u_s \times B).
\] (53)

Since $u_s = u_z z$, the continuity equation is trivially satisfied. Thus, the equation to be satisfied is
\[
\frac{dp_s}{dx} + n_s q_s u_{sz} B_y = 0.
\] (54)

By substituting $n$ and $B_y$ and by using the relation $p_s = n_s T_s$, we get
\[
\frac{dT_s}{dx} = 0.
\] (55)

If $T$ of both species are uniform, the solution satisfies the steady two fluid equation.

**Guide field**

We impose the external guid field in the $z$ direction, $B_{ex} = B_G z$, and $A_{ex} = B_G x y$. This alters the particle Hamiltonian,
\[
H_G = \frac{P_z^2}{2m} + (P_y - qB_G x)^2 + (P_z - qA_z^2) + q\phi.
\] (56)

The distribution function satisfying the Vlasov equation changes slightly to include $u_y A_y$ term,
\[
f_G = \frac{n}{\pi^{3/2} v_{th}^3} \exp \left[ -\left( \frac{v - u}{v_{th}} \right)^2 \right] \exp \left[ q( u_y A_y + u_z A_z) \right].
\] (57)

The additional field equation should be considered is the Ampère’s law in the $y$ direction. However, since $d^2 A_y/dx^2 = 0$, we can still take $u_y = 0$. Thus, we add an arbitrary constant guide field without changing the solution without a guide field.
Summary

The solution derived here is called the Harris sheet [Harris1962]. The solution is expressed as

\[ f_s(x, v_x, v_y, v_z) = \frac{n_0}{\pi^{3/2} v_{th}} \cosh^{-2} \left( \frac{x}{a} \right) \exp \left[ -\frac{v_x^2 + v_y^2 + (v_z - u_{sz})^2}{v_{th}^2} \right] . \]  

(58)

\( n_0 \) is the number density which is common for species to satisfy the neutrality. \( v_{th} = \sqrt{2T_s/m_s} \) is the thermal speed, \( T_s \) and \( m_s \) is the temperature and the mass, \( u_{sz} \) is the bulk flow in the \( z \) direction satisfying

\[ q_i u_{iz} = \frac{q_e u_{ez}}{T_i} . \]  

(59)

\( T_s \) and \( u_{sz} \) are constants. The current sheet width is

\[ \frac{1}{a} = \frac{1}{\delta_i} \left( \frac{u_{iz}}{v_{thi}} \right) \left( 1 + \frac{T_e}{T_i} \right)^{1/2} . \]  

(60)

The magnetic field and the number density is given by

\[ B_y(x) = B_0 \tanh \left( \frac{x}{\alpha} \right) , \]  

(61)

\[ n_s(x) = n_0 \cosh^{-2} \left( \frac{x}{\alpha} \right) . \]  

(62)

The plasma beta \( \beta = 2\mu_0 n_0 (T_i + T_e)/B_0^2 = 1 \).

3 Gyrokinetic equilibrium

The gyrokinetic equation and the gyrokinetic field equations are written as

\[ \frac{\partial h_s}{\partial t} + V_s \frac{\partial h_s}{\partial Z} + \frac{1}{B_0} \left\{ \langle \chi \rangle R, h_s \right\} - \langle C(h_s) \rangle R = q_s \frac{f_{0s}}{T_{0s}} \frac{\partial \langle \chi \rangle R}{\partial t} , \]  

(63)

\[ \sum_s \left[ -\frac{q_s^2 n_0 \phi}{T_{0s}} + q_s \int \langle h_s \rangle v R d\mathbf{v} \right] = 0 , \]  

(64)

\[ \nabla_\perp^2 A_\parallel = -\frac{1}{\mu_0} \sum_s q_s \int \langle h_s \rangle v R d\mathbf{v} , \]  

(65)

\[ B_0 \nabla_\parallel \delta B_\parallel = -\frac{1}{\mu_0} \nabla_\perp \cdot \sum_s \int \langle m v_\perp v_\parallel h_s \rangle R d\mathbf{v} . \]  

(66)

Consider the consistent collisionless equilibrium in one dimension having only \( A_\parallel (\phi = \delta B_\parallel = 0) \). The equations to be satisfied are

\[ \left\{ \langle \chi \rangle R, h_s \right\} = 0 , \]  

(67)

\[ \int \langle h_s \rangle v R d\mathbf{v} = 0 , \]  

(68)

\[ \nabla_\perp^2 A_\parallel = -\frac{1}{\mu_0} \sum_s q_s \int \langle h_s \rangle v R d\mathbf{v} , \]  

(69)

\[ \int \langle m v_\perp v_\parallel h_s \rangle R d\mathbf{v} = 0 . \]  

(70)
It is trivial that any one dimensional solution satisfies the steady gyrokinetic equation. The distribution function
should only produce the first order moment in the parallel direction to balance \( A_k \). Then, the general solution
of \( h_s \) is given by a fluctuating part of the shifted Maxwellian,

\[
\begin{align*}
  h_s(R, V_\perp, V_\parallel) &= \frac{2f_{0s}}{v_{th,s}^2} u_{||,s} (X) V_\parallel, \\
  f_{0s}(v) &= \frac{n_{0s}}{\pi^{3/2} v_{th,s}^2} \exp \left[ -\frac{v^2}{v_{th,s}^2} \right],
\end{align*}
\]  

(71)

where \( u_{||} \) is chosen such that the current produces the desired magnetic field. As we saw in the discussion of the
Harris sheet, the pressure balance must involve only in the \( x \) (perpendicular) direction. (There is no gradient in
the parallel direction.) In the gyrokinetic version of equilibrium, the pressure balance is satisfied by balancing
the magnetic pressure of \( B_0 \) (the guide field) and the static pressure given by \( f_0 \) (whatever the profiles of them
are), and the fluctuating part of static pressure is zero.

### A Hamiltonian Dynamics

We start with the Lagrangean defined by

\[
L(x, v, t) = \frac{1}{2}mv^2 - q\phi(x) + qv \cdot A(x),
\]  

(72)

where \( v = dx/dt \). Consider the dynamics which minimizes the action,

\[
S = \int_{t_1}^{t_2} L(x, v, t) dt.
\]  

(73)

Taking the variation of \( S \) gives

\[
\delta S = \int \left( \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial v} \delta v \right) dt
\]

\[
= \int \left( \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial v} \frac{dx}{dt} \delta x \right) dt
\]

\[
= \left[ \frac{\partial L}{\partial v} \delta x \right] + \int \left( \frac{\partial L}{\partial x} \delta x - \frac{dx}{dt} \left( \frac{\partial L}{\partial v} \right) \delta x \right) dt
\]

\[
= - \int \frac{dx}{dt} \frac{\partial L}{\partial v} \delta x dt.
\]  

(74)

Thus, for any given \( \delta x \), the minimization of \( S \) yields the Euler-Lagrange equation,

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial v} \right) - \frac{\partial L}{\partial x} = 0.
\]  

(75)

By Substituting the Lagrangean, we get

\[
\frac{\partial L}{\partial x} = -q \frac{\partial \phi}{\partial x} + q \frac{\partial}{\partial x} (v \cdot A),
\]  

(76)

\[
\frac{\partial L}{\partial v} = mv + qA.
\]  

(77)
and
\[ m \frac{d\mathbf{v}}{dt} + q \frac{\partial A}{\partial t} + q \left( \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} \right) A + q \frac{\partial \phi}{\partial \mathbf{x}} - q \left( \mathbf{v} \cdot \frac{\partial A}{\partial \mathbf{x}} + \mathbf{v} \times \frac{\partial}{\partial \mathbf{x}} \times A \right) = 0. \] (78)

Writing \( \nabla = \partial/\partial \mathbf{x} \), and defining
\[ E = -\nabla \phi - \frac{\partial A}{\partial t} \] (79)
\[ B = \nabla \times A, \] (80)
we obtain
\[ m \frac{d\mathbf{v}}{dt} = q(E + \mathbf{v} \times B). \] (81)

We Legendre transform the Lagrangean to \( Q, P \) coordinates,
\[ H(Q, P, t) = P \cdot \frac{dQ}{dt} - L(x, \mathbf{v}, t) = \frac{(P - qA)^2}{2m} + q\phi \] (82)
where we call \( H \) the Hamiltonian and
\[ Q = x, \] (83)
\[ P = \frac{\partial L}{\partial \mathbf{v}} = mv + qA. \] (84)

The variation of \( H \) is
\[ \delta H = \frac{\partial H}{\partial Q} \delta Q + \frac{\partial H}{\partial P} \delta P \]
\[ = \delta P \cdot \frac{dQ}{dt} + P \cdot \frac{d\delta Q}{dt} - \delta L \]
\[ = \delta P \cdot \frac{dQ}{dt} + P \cdot \frac{d\delta Q}{dt} - \left( \frac{\partial L}{\partial \mathbf{x}} \delta x + \frac{\partial L}{\partial \mathbf{v}} \delta \mathbf{v} \right) \]
\[ = \delta P \cdot \frac{dQ}{dt} + P \cdot \frac{d\delta Q}{dt} \left( \frac{\partial L}{\partial \mathbf{x}} \delta x + P \cdot \frac{dx}{dt} \right) \]
\[ = \delta P \cdot \frac{dQ}{dt} - \frac{dL}{dt} \frac{\partial L}{\partial \mathbf{v}} \delta \mathbf{v} \]
\[ = \delta P \cdot \frac{dQ}{dt} - \frac{\partial L}{\partial \mathbf{v}} \delta \mathbf{v}. \] (85)

The minimization of \( H \) gives the Hamilton’s equation,
\[ \frac{dQ}{dt} = \frac{\partial H}{\partial P}, \] (86)
\[ \frac{dP}{dt} = -\frac{\partial H}{\partial Q}. \] (87)

If \( H \) does not include \( t \) explicitly (autonomous), then the conservation of \( H \) immediately follows,
\[ \frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{dQ}{dt} \frac{\partial H}{\partial Q} + \frac{dP}{dt} \frac{\partial H}{\partial P} = 0. \] (88)
Now we show that the Hamilton’s equation of motion is equivalent to the Newton’s equation of motion. The partial derivatives are calculated as follows,

\[
\frac{\partial H}{\partial P} = \frac{P - qA}{m} \\
\frac{\partial H}{\partial Q} = \frac{1}{m} \left[ (P - qA) \times \left( \frac{\partial}{\partial Q} \times (P - qA) \right) - \left( (P - qA) \cdot \frac{\partial}{\partial Q} \right) (P - qA) \right] + \frac{\partial \phi}{\partial Q}
\]

\[
= -qv(Q, P) \times \frac{\partial}{\partial Q} A(Q) + qv(Q, P) \cdot \frac{\partial}{\partial Q} A(Q)
\]

(89)

Noting that \(\frac{d}{dt} = \frac{\partial}{\partial t} + (v \cdot \frac{\partial}{\partial Q})\), we get

\[
\frac{dx}{dt} = v
\]

(90)

\[
m \frac{dv}{dt} = -q \frac{\partial \phi}{\partial Q} - q \frac{\partial A}{\partial t} + qv \times \frac{\partial}{\partial Q} A = q(E + v \times B).
\]

(91)

References