Bessel Function

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The Bessel's differential equation is given by

$$\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}u}{\mathrm{d}z}\right)+\left(1-\frac{\nu^2}{z^2}\right)u=\frac{\mathrm{d}^2u}{\mathrm{d}z^2}+\frac{1}{z}\frac{\mathrm{d}u}{\mathrm{d}z}+\left(1-\frac{\nu^2}{z^2}\right)u=0. \tag{1}$$

The Bessel functions are the solution to this equation. The definitions of the Bessel functions are given by

$$\text{Bessel} \qquad J_\nu(z)=\left(\frac{z}{2}\right)^\nu\sum_{n=0}^{\infty}\frac{(-1)^n(z/2)^{2n}}{n!\Gamma(\nu+n+1)} \quad (z\neq\text{negative real}) \tag{2}$$

$$\text{Neumann} \qquad N_\nu(z)=Y_\nu(z)=\frac{1}{\sin\nu\pi}\left[\cos\nu\pi J_\nu(z)-J_{-\nu}(z)\right] \quad (\nu\neq\text{integer}, z\neq\text{negative real}) \tag{3}$$

$$\begin{aligned}N_n(z)=Y_n(z)=&\frac{1}{\pi}\left[\frac{\partial J_\nu(z)}{\partial\nu}-(-1)^n\frac{\partial J_{-\nu}(z)}{\partial\nu}\right]\\
=&\frac{2}{\pi}J_n(z)\left(\gamma+\log\frac{z}{2}\right)\\
&-\frac{1}{\pi}\left(\frac{z}{2}\right)^n\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(n+k)!}\left(\frac{z}{2}\right)^{2k}\left[\sum_{m=1}^{k}\frac{1}{m}+\sum_{m=1}^{n+k}\frac{1}{m}\right]\\
&-\frac{1}{\pi}\left(\frac{z}{2}\right)^{-n}\sum_{r=0}^{n-1}\frac{(n-r-1)!}{r!}\left(\frac{z}{2}\right)^{2r} \quad (n=0,1,2,\cdots,z\neq\text{negative real})\\
&\text{where } \gamma=0.57721\ldots \text{ is the Euler's gamma,}\\
&\text{and the last term is replaced by 0 for } n=0.\end{aligned} \tag{4}$$

$$\text{Hankel (first kind)} \qquad H_\nu^{(1)}=J_\nu(z)+\mathrm{i}N_\nu(z) \tag{5}$$

$$\text{Hankel (second kind)} \qquad H_\nu^{(2)}=J_\nu(z)-\mathrm{i}N_\nu(z). \tag{6}$$

J_ν, N_ν, $H_\nu^{(1,2)}$ are also called the first, second, and the third kind of the cylindrical function, and they are collectively refered as the ν-th cylindrical or Bessel (in a broad sense) function.

Another expression for the Bessel function (of the first kind) having an integer parameter is

$$J_n(z)=\frac{\mathrm{i}^{-n}}{2\pi}\int_0^{2\pi}\cos n\theta e^{\mathrm{i}z\cos\theta}\mathrm{d}\theta. \tag{7}$$

Now we show this satisfies the Bessel's equation. J_n satisfies the following recurrence relation,

$$J_{n-1} + J_{n+1} = \frac{2n}{z} J_n. \tag{8}$$

Proof.

$$\begin{aligned}
z(J_{n-1} + J_{n+1}) =& z\frac{\mathrm{i}^{-n+1}}{2\pi}\int_0^{2\pi} \cos(n-1)\theta e^{\mathrm{i}z\cos\theta}\mathrm{d}\theta + z\frac{\mathrm{i}^{-n-1}}{2\pi}\int_0^{2\pi} \cos(n+1)\theta e^{\mathrm{i}z\cos\theta}\mathrm{d}\theta \\
=& \frac{\mathrm{i}^{-n}}{2\pi}\int_0^{2\pi} \mathrm{i}z(\cos(n-1)\theta - \cos(n+1)\theta)e^{\mathrm{i}z\cos\theta}\mathrm{d}\theta \\
=& \frac{\mathrm{i}^{-n}}{2\pi}\int_0^{2\pi} 2\mathrm{i}z\sin n\theta\sin\theta e^{\mathrm{i}z\cos\theta}\mathrm{d}\theta \\
=& -2\frac{\mathrm{i}^{-n}}{2\pi}\int_0^{2\pi} \sin n\theta\frac{\partial}{\partial\theta}\left(e^{\mathrm{i}z\cos\theta}\right)\mathrm{d}\theta \\
=& -2\frac{\mathrm{i}^{-n}}{2\pi}\left(\left[\sin n\theta e^{\mathrm{i}z\cos\theta}\right]_0^{2\pi} - \int_0^{2\pi} n\cos n\theta e^{\mathrm{i}z\cos\theta}\mathrm{d}\theta\right) \\
=& 2nJ_n.
\end{aligned}$$

□

Next, we evaluate the derivatives with respect to z.

$$\begin{aligned}
\frac{\mathrm{d}J_n}{\mathrm{d}z} =& \frac{\mathrm{i}^{-n}}{2\pi}\int_0^{2\pi} \mathrm{i}\cos\theta\cos n\theta e^{\mathrm{i}z\cos\theta}\mathrm{d}\theta \\
=& \frac{\mathrm{i}^{-n}}{2\pi}\int_0^{2\pi} \frac{\mathrm{i}}{2}(\cos(n+1)\theta + \cos(n-1)\theta)e^{\mathrm{i}z\cos\theta}\mathrm{d}\theta \\
=& \frac{1}{2}\frac{\mathrm{i}^{-n+1}}{2\pi}\frac{\mathrm{i}^2}{\mathrm{i}^2}\int_0^{2\pi} \cos(n+1)\theta e^{\mathrm{i}z\cos\theta}\mathrm{d}\theta + \frac{1}{2}\frac{\mathrm{i}^{-n+1}}{2\pi}\int_0^{2\pi} \cos(n-1)\theta e^{\mathrm{i}z\cos\theta}\mathrm{d}\theta \\
=& -\frac{1}{2}J_{n+1} + \frac{1}{2}J_{n-1} = -\frac{1}{2}(\frac{2n}{z}J_n - J_{n-1}) + \frac{1}{2}J_{n-1} = J_{n-1} - \frac{n}{z}J_n, \qquad (9) \\
\frac{\mathrm{d}^2J_n}{\mathrm{d}z^2} =& \frac{\mathrm{d}J_{n-1}}{\mathrm{d}z} - \frac{n}{z}\frac{\mathrm{d}J_n}{\mathrm{d}z} + \frac{n}{z^2}J_n \\
=& \left(J_{n-2} - \frac{n-1}{z}J_{n-1}\right) - \frac{n}{z}\left(J_{n-1} - \frac{n}{z}J_n\right) + \frac{n}{z^2}J_n \\
=& \left(-J_n + \frac{2(n-1)}{z}J_{n-1}\right) - \frac{2n-1}{2}J_{n-1} + \frac{n^2+n}{z^2}J_n \\
=& -\frac{1}{z}J_{n-1} + \left(\frac{n^2+n}{z^2} - 1\right)J_n \\
=& -\frac{1}{z}\left(\frac{\mathrm{d}J_n}{\mathrm{d}z} + \frac{n}{z}J_n\right) + \left(\frac{n^2+n}{z^2} - 1\right)J_n \\
=& -\frac{1}{z}\frac{\mathrm{d}J_n}{\mathrm{d}z} + \left(\frac{n^2}{z^2} - 1\right)J_n. \qquad (10)
\end{aligned}$$

Thus, J_n satisfies the Bessel's equation.

(7) can be derived from the generating function expression of the Bessel function

$$e^{\mathrm{i}z\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(z)e^{\mathrm{i}n\theta}. \tag{11}$$

By replacing θ by $-\theta$, it is easily expanded to

$$e^{\pm\mathrm{i}z\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(z)e^{\pm\mathrm{i}n\theta}. \tag{12}$$

We also use a transform $\theta' = \theta \pm \pi/2$ to obtain the expression for $\exp(\pm\mathrm{i}z\cos\theta)$. Plugging the relations

$$\begin{aligned}\sin\theta =& \sin(\theta' \mp \frac{\pi}{2}) = \mp\cos\theta'\sin\frac{\pi}{2} = \mp\cos\theta', \\ e^{\mathrm{i}n\theta} =& e^{\mathrm{i}n(\theta'\mp\pi/2)} = e^{\mp\mathrm{i}n\pi/2}e^{\mathrm{i}n\theta'} = (\mp\mathrm{i})^n e^{\mathrm{i}n\theta'}\end{aligned} \tag{13}$$

into (11), we get

$$e^{\mathrm{i}z\sin\theta} = e^{\mp\mathrm{i}z\cos\theta'} = \sum_{n=-\infty}^{\infty} J_n(z)(\mp\mathrm{i})^n e^{\mathrm{i}n\theta'}. \tag{14}$$

(This is called the Jacobi-Anger expansion.) Thus, we have the following relations,

$$e^{\pm\mathrm{i}z\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(z)e^{\pm\mathrm{i}n\theta}, \quad e^{\pm\mathrm{i}z\cos\theta} = \sum_{n=-\infty}^{\infty} (\pm\mathrm{i})^n J_n(z)e^{\mathrm{i}n\theta}. \tag{15}$$

Consider the integral

$$\frac{1}{2\pi}\int_0^{2\pi} \cos n\theta e^{\mathrm{i}z\cos\theta}\mathrm{d}\theta. \tag{16}$$

Substituting the above expression yields

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \cos n\theta e^{\mathrm{i}z\cos\theta}\mathrm{d}\theta =& \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{\mathrm{i}n\theta}+e^{-\mathrm{i}n\theta}}{2}\sum_{l=-\infty}^{\infty}\mathrm{i}^l J_l(z)e^{\mathrm{i}l\theta}\mathrm{d}\theta \\ =& \sum_{l=\infty}^{\infty}\frac{\mathrm{i}^l}{2}J_l(z)\frac{1}{2\pi}\int_0^{2\pi}\left(e^{\mathrm{i}(n+l)\theta}+e^{\mathrm{i}(-n+l)\theta}\right)\mathrm{d}\theta \\ =& \frac{\mathrm{i}^n}{2}J_n(z)+\frac{\mathrm{i}^{-n}}{2}J_{-n}(z) \\ =& \frac{\mathrm{i}^n}{2}J_n(z)\left(1+(-1)^n\mathrm{i}^{-2n}\right) = \mathrm{i}^n J_n(z),\end{aligned} \tag{17}$$

wheere $J_{-n}(z) = (-1)J_n(z)$ is used. Thus, we obtain

$$J_n(z) = \frac{\mathrm{i}^{-n}}{2\pi}\int_0^{2\pi}\cos n\theta e^{\mathrm{i}z\cos\theta}\mathrm{d}\theta \tag{18}$$

It is straightforwardly calculated that

$$\frac{1}{2\pi}\int_0^{2\pi}\cos n\theta e^{-\mathrm{i}z\cos\theta}\mathrm{d}\theta=(-\mathrm{i})^nJ_n(z). \tag{19}$$

Thus,

$$J_n(z)=\frac{(\pm\mathrm{i})^{-n}}{2\pi}\int_0^{2\pi}\cos n\theta e^{\pm\mathrm{i}z\cos\theta}\mathrm{d}\theta. \tag{20}$$

Next, we consider

$$\frac{1}{2\pi}\int_0^{2\pi}\sin n\theta e^{\pm\mathrm{i}z\cos\theta}\mathrm{d}\theta. \tag{21}$$

It is calculated as

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\sin n\theta e^{\pm\mathrm{i}z\cos\theta}\mathrm{d}\theta=&\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{\mathrm{i}n\theta}-e^{-\mathrm{i}n\theta}}{2\mathrm{i}}\sum_{l=-\infty}^{\infty}(\pm\mathrm{i})^lJ_l(z)e^{\mathrm{i}l\theta}\mathrm{d}\theta\\ =&\sum_{l=-\infty}^{\infty}\frac{(\pm\mathrm{i})^l}{2\mathrm{i}}J_l(z)\frac{1}{2\pi}\int_0^{2\pi}\left(e^{\mathrm{i}(n+l)\theta}-e^{\mathrm{i}(-n+l)\theta}\right)\mathrm{d}\theta\\ =&-\frac{(\pm\mathrm{i})^n}{2\mathrm{i}}J_n(z)+\frac{(\pm\mathrm{i})^{-n}}{2\mathrm{i}}J_{-n}(z)\\ =&\frac{(\pm\mathrm{i})^n}{2\mathrm{i}}J_n(z)\left(-1+(\pm\mathrm{i})^{-2n}(-1)^n\right)=0.\end{aligned} \tag{22}$$

Other combinations are

$$\frac{1}{2\pi}\int_0^{2\pi}\cos n\theta e^{\pm\mathrm{i}z\sin\theta}\mathrm{d}\theta,\quad\frac{1}{2\pi}\int_0^{2\pi}\sin n\theta e^{\pm\mathrm{i}z\cos\theta}\mathrm{d}\theta. \tag{23}$$

Each calculated as follows,

$$
\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi}\cos n\theta e^{\pm \mathrm{i}z\sin\theta}\mathrm{d}\theta =& \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{\mathrm{i}n\theta}+e^{-\mathrm{i}n\theta}}{2}\sum_{l=-\infty}^{\infty}J_l(z)e^{\pm\mathrm{i}l\theta}\mathrm{d}\theta \\
=& \sum_{l=-\infty}^{\infty}\frac{1}{2}J_l(z)\frac{1}{2\pi}\int_0^{2\pi}\left(e^{\mathrm{i}(n\pm l)\theta}+e^{\mathrm{i}(-n\pm l)\theta}\right)\mathrm{d}\theta \\
=& \frac{1}{2}J_{\mp n}(z)+\frac{1}{2}J_{\pm n}(z)=\frac{1}{2}(1+(-1)^n)J_n(z) \\
=& \begin{cases} J_n(z) & n=\text{even} \\ 0 & n=\text{odd} \end{cases},
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi}\sin n\theta e^{\pm \mathrm{i}z\sin\theta}\mathrm{d}\theta =& \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{\mathrm{i}n\theta}-e^{-\mathrm{i}n\theta}}{2\mathrm{i}}\sum_{l=-\infty}^{\infty}J_l(z)e^{\pm\mathrm{i}l\theta}\mathrm{d}\theta \\
=& \sum_{l=-\infty}^{\infty}\frac{1}{2\mathrm{i}}J_l(z)\frac{1}{2\pi}\int_0^{2\pi}\left(e^{\mathrm{i}(n\pm l)\theta}-e^{\mathrm{i}(-n\pm l)\theta}\right)\mathrm{d}\theta \\
=& -\frac{\mathrm{i}}{2}J_{\mp n}(z)+\frac{\mathrm{i}}{2}J_{\pm n}(z)=\pm\frac{\mathrm{i}}{2}(1-(-1)^n)J_n(z) \\
=& \begin{cases} 0 & n=\text{even} \\ \pm\mathrm{i}J_n(z) & n=\text{odd} \end{cases}.
\end{aligned} \tag{25}
$$

We can combine these results in the following way,

$$
\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi}e^{\mathrm{i}n\theta}e^{\pm z\sin\theta}\mathrm{d}\theta =& \frac{1}{2\pi}\int_0^{2\pi}(\cos n\theta+\mathrm{i}\sin n\theta)e^{\pm z\sin\theta}\mathrm{d}\theta \\
=& \begin{cases} J_n(z) & n=\text{even} \\ \mp J_n(z) & n=\text{odd} \end{cases} \\
=& (\mp 1)^n J_n(z).
\end{aligned} \tag{26}
$$

Or, we can write

$$
J_n(z)=\frac{(\mp 1)^{-n}}{2\pi}\int_0^{2\pi}e^{\mathrm{i}n\theta}e^{\pm\mathrm{i}z\sin\theta}\mathrm{d}\theta. \tag{27}
$$

Summary

Here, we summarize the relations.

$$e^{\pm \mathrm{i}z\sin\theta} = \sum_n J_n e^{\mathrm{i}n\theta} \qquad e^{\pm \mathrm{i}z\cos\theta} = \sum_n (\pm \mathrm{i})^n J_n e^{\mathrm{i}n\theta} \tag{28}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \cos n\theta e^{\pm \mathrm{i}z\cos\theta} \mathrm{d}\theta = (\pm \mathrm{i})^n J_n(z) \qquad J_n(z) = \frac{(\pm \mathrm{i})^{-n}}{2\pi}\int_0^{2\pi} \cos n\theta e^{\pm \mathrm{i}z\cos\theta}\mathrm{d}\theta \tag{29}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \sin n\theta e^{\pm \mathrm{i}z\cos\theta} \mathrm{d}\theta = 0 \tag{30}$$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\mathrm{i}n\theta} e^{\pm \mathrm{i}z\cos\theta} = (\pm \mathrm{i})^n J_n(z) \qquad J_n(z) = \frac{(\pm \mathrm{i})^{-n}}{2\pi}\int_0^{2\pi} e^{\mathrm{i}n\theta} e^{\pm \mathrm{i}z\cos\theta}\mathrm{d}\theta \tag{31}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \cos n\theta e^{\pm \mathrm{i}z\sin\theta} \mathrm{d}\theta = \begin{cases} J_n(z) & n = \text{even} \\ 0 & n = \text{odd} \end{cases} \tag{32}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \sin n\theta e^{\pm \mathrm{i}z\sin\theta} \mathrm{d}\theta = \begin{cases} 0 & n = \text{even} \\ \pm \mathrm{i} J_n(z) & n = \text{odd} \end{cases} \tag{33}$$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\mathrm{i}n\theta} e^{\pm \mathrm{i}z\sin\theta} \mathrm{d}\theta = (\mp 1)^n J_n(z) \qquad J_n(z) = \frac{(\mp 1)^{-n}}{2\pi}\int_0^{2\pi} e^{\mathrm{i}n\theta} e^{\pm \mathrm{i}z\sin\theta}\mathrm{d}\theta \tag{34}$$

Modified Bessel functions

By giving pure imaginary argument to the Bessel functions, we obtain the modified Bessel functions.

$$\text{First kind} \qquad I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{n!\Gamma(\nu + n + 1)} \tag{35}$$

$$= \begin{cases} e^{-\mathrm{i}\nu\pi/2} J_\nu\left(e^{\mathrm{i}\pi/2} z\right) & (-\pi < \arg z < \pi/2) \\ e^{3\mathrm{i}\nu\pi/2} J_\nu\left(e^{-3\mathrm{i}\pi/2} z\right) & (\pi/2 < \arg z < \pi) \end{cases} \quad (z \neq \text{negative real}) \tag{36}$$

$$\text{Second kind} \qquad K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi} = \frac{\mathrm{i}\pi}{2} e^{\mathrm{i}\nu\pi/2} H_\nu^{(1)}(\mathrm{i}z) = \frac{\mathrm{i}\pi}{2} e^{-\mathrm{i}\nu\pi/2} H_{-\nu}^{(1)}(\mathrm{i}z) \tag{37}$$

$$(\nu \neq \text{integer}, z \neq \text{negative real}), \tag{38}$$

$$K_n(z) = K_{-n}(z) = \frac{(-1)^n}{2}\left[\frac{\partial I_{-\nu}(z)}{\partial \nu} - \frac{\partial I_\nu(z)}{\partial \nu}\right]_{\nu = n} \tag{39}$$

$$= (-1)^{n+1} I_n(z)\left(\gamma + \log\frac{z}{2}\right) \tag{40}$$

$$+ \frac{(-1)^n}{2}\sum_{k=0}^{\infty} \frac{(z/2)^{n+2k}}{k!(n+k)!}\left[\sum_{m=1}^{k}\frac{1}{m} + \sum_{m=1}^{k+n}\frac{1}{m}\right] \tag{41}$$

$$+ \frac{1}{2}\sum_{r=0}^{n-1}(-1)^r \frac{(n-r-1)!}{r!}\left(\frac{r}{2}\right)^{2r-n}, \text{the last term is replaced by 0 for } n = 0. \tag{42}$$

These modified Bessel functions satisty the modified Bessel's equation,

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} - \left(1 + \frac{\nu^2}{z^2}\right) w = 0. \tag{43}$$

The generating function expression of I_n is given by

$$\exp\left[\frac{x}{2}\left(t+\frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} I_n(z)t^n, \tag{44}$$

or by replacing t by $e^{\mathrm{i}\theta}$

$$e^z \cos\theta = \sum_{n=-\infty}^{\infty} I_n(z)e^{\mathrm{i}n\theta}. \tag{45}$$

Integrals

$$\int_0^\infty e^{-a^2x^2} x J_0(bx)\mathrm{d}x = \frac{e^{-b^2/(4a^2)}}{2a^2} \quad \text{(Weber)} \tag{46}$$

$$\int_0^\infty e^{-a^2x^2} x^2 J_1(bx)\mathrm{d}x = \frac{be^{-b^2/(4a^2)}}{4a^4} \tag{47}$$

$$\int_0^\infty e^{-a^2x^2} x J_n^2(bx)\mathrm{d}x = \frac{e^{-b^2/(2a^2)}}{2a^2} I_n\left(\frac{b^2}{2a^2}\right) \tag{48}$$